MULTIPLE HAMILTONIAN STRUCTURE OF BOGOYAVLENSKY-TODA LATTICES

Pantelis A. Damianou

Department of Mathematics and Statistics
University of Cyprus
P. O. Box 20537, 1678 Nicosia, Cyprus
Email: damianou@ucy.ac.cy

ABSTRACT

This paper is mainly a review of the multi–Hamiltonian nature of Toda and generalized Toda lattices corresponding to the classical simple Lie groups but it includes also some new results. The areas investigated include master symmetries, recursion operators, higher Poisson brackets, invariants and group symmetries for the systems. In addition to the positive hierarchy we also consider the negative hierarchy which is crucial in establishing the bi–Hamiltonian structure for each particular simple Lie group. Finally, we include some results on point and Noether symmetries and an interesting connection with the exponents of simple Lie groups. The case of exceptional simple Lie groups is still an open problem.

Mathematics Subject Classification: 37J35, 22E70 and 70H06.

Key words Toda lattice, Poisson brackets, master symmetries, bi-Hamiltonian systems, group symmetries, simple Lie groups.

CONTENTS

1 INTRODUCTION

In this paper we review the bi–Hamiltonian and multiple Hamiltonian nature of the Toda lattices corresponding to simple Lie groups. These are systems that generalize the usual finite, non–periodic Toda lattice (which corresponds to a root system of type A_n). This generalization is due to Bogoyavlensky [1]. These systems were studied extensively in [2] where the solution of the system was connected intimately with the representation theory of simple Lie groups. There are also studies by Olshanetsky and Perelomov [3] and Adler, van Moerbeke [4]. We will call such systems the Bogoyavlensky-Toda lattices.
We begin with the following more general definition which involves systems with exponential interaction: Consider a Hamiltonian of the form

$$H = \frac{1}{2}(\mathbf{p}, \mathbf{p}) + \sum_{i=1}^{m} e^{(\mathbf{v}_i\, , \mathbf{q})} \ , \tag{1}$$

where $\mathbf{q} = (q_1, \ldots, q_N)$, $\mathbf{p} = (p_1, \ldots, p_N)$, $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are vectors in $\mathbf{R}^N$ and $(\ ,\ )$ is the standard inner product in $\mathbf{R}^N$. The set of vectors $\Delta = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is called the spectrum of the system.
In this paper we limit our attention to the case where the spectrum is a system of simple roots for a simple Lie algebra $\mathcal{G}$. In this case $m = l = \operatorname{rank} \mathcal{G}$. It is worth mentioning that the case where m, N are arbitrary is an open and unexplored area of research. The main exception is the work of Kozlov and Treshchev [5] where a classification of system (1) is performed under the assumption that the system possesses N polynomial (in the momenta) integrals. We also note the papers by Ranada [6] and Annamalai, Tamizhmani [7]. Such systems are called Birkhoff integrable. For each Hamiltonian in (1) we associate a Dynkin type diagram as follows: It is a graph whose vertices correspond to the elements of Δ. Each pair of vertices v_i, v_j are connected by

$$\frac{4(v_i, v_j)^2}{(v_i, v_i)(v_j, v_j)}$$

edges.

Example: The usual Toda lattice corresponds to a Lie algebra of type A_{N-1}. In other words $m = l = N - 1$ and we choose Δ to be the set:

$$\mathbf{v}_1 = (1, -1, 0, \ldots, 0), \ldots\ldots\ldots, \mathbf{v}_{N-1} = (0, 0, \ldots, 0, 1, -1) \ .$$

The graph is the usual Dynkin diagram of a Lie algebra of type A_{N-1}. The Hamiltonian becomes

$$H(q_1, \ldots, q_N,\, p_1, \ldots, p_N) = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + \sum_{i=1}^{N-1} e^{q_i - q_{i+1}} \ , \tag{2}$$

which is the well-known classical, non–periodic Toda lattice.
It is more convenient to work, instead in the space of the natural (q, p) variables, with the Flaschka variables (a, b) which are defined by:

$$\begin{array}{rcll} a_i & = & \frac{1}{2} e^{\frac{1}{2}(\mathbf{v}_i, \mathbf{q})} & \quad i = 1, 2, \ldots, m \\ b_i & = & -\frac{1}{2} p_i & \quad\quad i = 1, 2, \ldots, N \ . \end{array} \tag{3}$$

We end–up with a new set of polynomial equations in the variables (a, b). One can write the equations in Lax pair form (at least this is well–known for spectrum corresponding to simple Lie algebras); see for example [8]. The Lax pair $(L(t), B(t))$ in $\mathcal{G}$ can be described in terms of the root system as follows:

$$L(t) = \sum_{i=1}^{l} b_i(t) h_{\alpha_i} + \sum_{i=1}^{l} a_i(t)(x_{\alpha_i} + x_{-\alpha_i}) \ ,$$

$$B(t) = \sum_{i=1}^{l} a_i(t)(x_{\alpha_i} - x_{-\alpha_i}) \ .$$

As usual h_{α_i} is an element of a fixed Cartan subalgebra and x_{α_i} is a root vector corresponding to the simple root α_i. The Chevalley invariants of $\mathcal{G}$ provide for the constants of motion. We will describe them separately for each case.

In this paper we begin with a review of the A_N Toda system. The theory and bi–Hamiltonian structure for this case is well–developed and the results are well–known. We present a review of the results in sections 3–5 and then, for the remaining part of the paper, we deal exclusively with the other classical simple Lie algebras of type B_N, C_N and D_N. We will demonstrate that these systems are bi–Hamiltonian and then illustrate with some small dimensional examples, namely B_2, C_3 and D_4.

The multi–Hamiltonian structure of the Toda lattice was established in [9]. For the remaining Bogoyavlensky–Toda systems it was established recently in several papers. The results for B_N Bogoyavlensky-Toda lattice were computed in [10] in Flaschka coordinates, and in [11] in (q, p) coordinates. The C_N case is in [12] in Flaschka coordinates, and [11] in natural (q, p) coordinates. The D_N case was settled in [13]. The bi–Hamiltonian structure of these systems was established recently in [14]. The negative Toda hierarchy was constructed in [15] and it was crucial in establishing the bi–Hamiltonian formulation in [14].

The construction of the bi–Hamiltonian pair may be summarized as follows:

Define a recursion operator $\mathcal{R}$ in (a, b) space by finding a second bracket, π_3, and inverting the initial Poisson bracket π_1. Define the negative recursion operator $\mathcal{N}$ by inverting the second Poisson bracket π_3. This recursion operator is the inverse of the operator $\mathcal{R}$. Finally, define a new rational bracket π_{-1} by $\pi_{-1} = \mathcal{N}\pi_1 = \pi_1 \pi_3^{-1} \pi_1$. We obtain a bi–Hamiltonian formulation of the system:

$$\pi_1 \nabla H_2 = \pi_{-1} \nabla H_4 \ ,$$

where $H_i = \frac{1}{i} \mathrm{tr}\, L^i$. The brackets π_1 and π_{-1} are compatible and Poisson.

There is also an interesting connection with the exponents of the corresponding Lie group. For example, in the case of D_N there is a sequence of invariants $H_2, H_4, \ldots,$ of even degree and an additional invariant of degree N. Let χ_i denote the Hamiltonian vector field generated by H_i and let Z_0 denote a conformal symmetry. Then we have

$$[Z_0, \chi_j] = f(j) \chi_j \ .$$

The values of $f(j)$ corresponding to independent χ_j generate all the exponents except one. When Z_0 acts on the Hamiltonian vector field χ_P, where P is the invariant corresponding to the Pfaffian of the Jacobi matrix, we obtain the last exponent $N-1$. For example, in the case of D_5 the exponents are 1, 3, 5, 7, and 4. The independent invariants are H_2, H_4, H_6, H_8 and P_5 where $H_{2i} = \frac{1}{2i}$ Tr L^{2i} and $P_5 = \sqrt{\det L}$. We obtain

$$
\begin{array}{rcl}
[Z_0, \chi_2] & = & \chi_2 \\
[Z_0, \chi_4] & = & 3\chi_4 \\
[Z_0, \chi_6] & = & 5\chi_6 \\
[Z_0, \chi_8] & = & 7\chi_8 \\
[Z_0, \chi_{P_5}] & = & 4\chi_{P_5} \ .
\end{array}
$$

In other words, the coefficients on the right hand side are precisely the exponents of a simple Lie group of type D_5.

2 BACKGROUND

In this section we review the necessary background from Poisson and symplectic geometry, bi–Hamiltonian systems, master symmetries and recursion operators.

2.1 Schouten bracket

We list some properties of the Schouten bracket following [16], [17], [18]. Let M be a C^∞ manifold, $N = C^\infty(M)$ the algebra of C^∞ real–valued functions on M. A contravariant, antisymmetric tensor of order p will be called a p-tensor for short. These tensors form a superspace endowed with a Lie–superalgebra structure via the Schouten bracket.

The Schouten bracket assigns to each p-tensor A, and q-tensor B, a $(p+q-1)$-tensor, denoted by $[A, B]$. For $p = 1$ we have $[A, B] = L_A B$ where L_A is the Lie-derivative in the direction of the vector field A. The bracket satisfies

i)

$$[A, B] = (-1)^{pq}[B, A] \ . \tag{4}$$

ii) If C is a r-tensor

$$(-1)^{pq}[[B, C], A] + (-1)^{qr}[[C, A], B] + (-1)^{rp}[[A, B], C] = 0 \ . \tag{5}$$

iii)

$$[A, B \wedge C] = [A, B] \wedge C + (-1)^{pq+q} B \wedge [A, C] \ . \tag{6}$$

2.2 Poisson Manifolds

We review the basic definitions and properties of Poisson manifolds following [16], [18], [19]. A Poisson structure on M is a bilinear form, called the Poisson bracket $\{\ ,\ \} : N \times N \to N$ such that

i)

$$\{f, g\} = -\{g, f\} \tag{7}$$

ii)

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \tag{8}$$

iii)

$$\{f, gh\} = \{f, g\}h + \{f, h\}g \ . \tag{9}$$

Properties *i)* and *ii)* define a Lie algebra structure on N. *ii)* is called the Jacobi identity and *iii)* is the analogue of Leibniz rule from calculus. A Poisson manifold is a manifold M together with a Poisson bracket $\{\ ,\ \}$.

To a Poisson bracket one can associate a 2-tensor π such that

$$\{f, g\} = \langle \pi, df \wedge dg \rangle \ . \tag{10}$$

Jacobi's identity is equivalent to the condition $[\pi, \pi] = 0$ where $[\ ,\ ]$ is the Schouten bracket. Therefore, one could define a Poisson manifold by specifying a pair (M, π) where M is a manifold and π a 2-tensor satisfying $[\pi, \pi] = 0$. In local coordinates $(x_1, x_2, \ldots, x_n)$, π is given by

$$\pi = \sum_{i,j} \pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \tag{11}$$

and

$$\{f, g\} = \langle \pi, df \wedge dg \rangle = \sum_{i,j} \pi_{ij} \frac{\partial f}{\partial x_i} \wedge \frac{\partial g}{\partial x_j} \ . \tag{12}$$

In particular $\{x_i, x_j\} = \pi_{ij}(x)$. Knowledge of the Poisson matrix (π_{ij}) is sufficient to define the bracket of arbitrary functions. The rank of the matrix (π_{ij}) at a point $x \in M$ is called the rank of the Poisson structure at x.

A function $F : M_1 \to M_2$ between two Poisson manifolds is called a Poisson mapping if

$$\{f \circ F, g \circ F\}_1 = \{f, g\}_2 \circ F \tag{13}$$

for all $f, g \in C^\infty(M_2)$. In terms of tensors, $F_* \pi_1 = \pi_2$. Two Poisson manifolds are called isomorphic, if there exists a diffeomorphism between them which is a Poisson mapping.

The Poisson bracket allows one to associate a vector field to each element $f \in N$. The vector field χ_f is defined by the formula

$$\chi_f(g) = \{f, g\} \ . \tag{14}$$

It is called the Hamiltonian vector field generated by f. In terms of the Schouten bracket

$$\chi_f = [\pi, f] \ . \tag{15}$$

Hamiltonian vector fields are infinitesimal automorphisms of the Poisson structure. These are vector fields X satisfying $L_X \pi = 0$. In the case of Hamiltonian vector fields we have

$$L_{\chi_f} \pi = [\pi, \chi_f] = [\pi, [\pi, f]] = -2[[\pi, \pi], f] = 0 \ . \tag{16}$$

The Hamiltonian vector fields form a Lie algebra and in fact

$$[\chi_f, \chi_g] = \chi_{\{f,g\}} \ . \tag{17}$$

So, the map $f \to \chi_f$ is a Lie algebra homomorphism.

The Poisson structure defines a bundle map

$$\pi^* : T^*M \to TM \tag{18}$$

such that

$$\pi^*(df) = \chi_f \ . \tag{19}$$

The rank of the Poisson structure at a point $x \in M$ is the rank of $\pi^*_x : T^*_x M \to T_x M$. Throughout this paper we use the symbol π to denote a Poisson tensor but occasionally we use the same symbol to denote the matrix of the components of the tensor (i.e. the Poisson matrix). The same convention applies for the recursion operators.

The functions in the center of N are called Casimirs. It is the set of functions f so that $\{f, g\} = 0$ for all $g \in N$. These are functions which are constant along the orbits of Hamiltonian vector fields. The differentials of these functions are in the kernel of π^*. In terms of the Schouten bracket a Casimir satisfies $[\pi, f] = 0$.

Given a function f, there is a reasonable algorithm for constructing a Poisson bracket in which f is a Casimir. One finds two vector fields X_1 and X_2 such that $L_{X_1} f = L_{X_2} f = 0$. If in addition X_1, X_2 and $[X_1, X_2]$ are linearly dependent, then $X_1 \wedge X_2$ is a Poisson tensor and f is a Casimir in this bracket. In fact

$$[f, X_1 \wedge X_2] = [f, X_1] \wedge X_2 - X_1 \wedge [f, X_2] = 0 \ . \tag{20}$$

More generally, there is a formula due to Flaschka and Ratiu which gives locally a Poisson bracket when the number of Casimirs is 2 less than the dimension of the space. Let $f_1, f_2, \ldots, f_r$ be functions on $\mathbf{R}^{r+2}$.
Then the formula

$$\omega\{g, h\} = df_1 \wedge \ldots \wedge df_r \wedge dg \wedge dh \tag{21}$$

where ω is a non-vanishing $r + 2$ form, defines a Poisson bracket on $\mathbf{R}^{r+2}$ and the functions $f_1, \ldots, f_r$ are Casimirs. For more details on this formula, see [20].

Multiplication of a Poisson bracket by a Casimir gives another Poisson bracket. Suppose $[\pi, \pi] = 0$ and $[\pi, f] = 0$. Then

$$[f\pi, f\pi] = f \wedge [f, \pi] \wedge \pi + f \wedge \pi \wedge [\pi, f] + f^2[\pi, \pi] = 0 \ . \tag{22}$$

2.3 Symplectic and Lie Poisson manifolds

The most basic examples of Poisson brackets are the symplectic and Lie-Poisson brackets.

i) **Symplectic manifolds:** A *symplectic manifold* is a pair $(M^{2n},\ \omega)$ where M^{2n} is an even dimensional manifold and ω is a closed, non-degenerate 2–form. The associated isomorphism

$$\mu : TM \to T^*M \tag{23}$$

extends naturally to a tensor bundle isomorphism still denoted by μ. Let $\lambda = \mu^{-1}$, $f \in N$ and let $\chi_f = \lambda(df)$ be the corresponding Hamiltonian vector field. The symplectic bracket is given by

$$\{f, g\} = \omega(\chi_f, \chi_g) \ . \tag{24}$$

In the case of $\mathbf{R}^{2n}$, according to a Theorem of Darboux, there are coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$, so that

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i \tag{25}$$

and the Poisson bracket is the standard symplectic bracket on $\mathbf{R}^{2n}$.

ii) **Lie–Poisson :** Let $M = \mathcal{G}^*$ where $\mathcal{G}$ is a Lie algebra. For $a \in \mathcal{G}$, define the function Φ_a on $\mathcal{G}^*$ by

$$\Phi_a(\mu) = \langle a, \mu \rangle \tag{26}$$

where $\mu \in \mathcal{G}^*$ and $\langle\ ,\ \rangle$ is the pairing between $\mathcal{G}$ and $\mathcal{G}^*$. Define a bracket on $\mathcal{G}^*$ by

$$\{\Phi_a, \Phi_b\} = \Phi_{[a,b]}\ . \tag{27}$$

This bracket is easily extended to arbitrary C^∞ functions on $\mathcal{G}^*$. The bracket of linear functions is linear and every linear bracket is of this form, i.e., it is associated with a Lie algebra. Therefore, the classification of linear Poisson brackets is equivalent to the classification of Lie algebras.

2.4 Local theory

In his paper [19] A. Weinstein proved the so-called "splitting theorem", which describes the local behavior of Poisson manifolds.

Theorem 1 *Let x_0 be a point in a Poisson manifold M. Then near x_0, M is isomorphic to a product $S \times N$ where S is symplectic, N is a Poisson manifold, and the rank of N at x_0 is zero.*

S is called the symplectic leaf through x_0 and N is called the transverse Poisson structure at x_0. N is unique up to isomorphism. So, through each point x_0 passes a symplectic leaf S_{x_0} whose dimension equals the rank of the Poisson structure on M at x_0. The bracket on the transverse manifold N_{x_0} can be calculated using Dirac's constraint bracket formula.

Theorem 2 *Let x_0 be a point in a Poisson manifold M and let U be a neighborhood of x_0 which is isomorphic to a product $S \times N$ as in Weinstein's splitting Theorem. Let p_i, $i = 1, \ldots, 2n$ be functions on U such that*

$$N = \{x \in U \,|\, p_i(x) = \text{constant}\}\ . \tag{28}$$

Denote by $P = P_{ij} = \{p_i, p_j\}$ and by P^{ij} the inverse matrix of P. Then the bracket formula for the transverse Poisson structure on N is given as follows:

$$\{F, G\}_N(x) = \{\hat{F}, \hat{G}\}_M(x) + \sum_{i,j}^{2n} \{\hat{F}, p_i\}_M(x) P^{ij}(x) \{\hat{G}, p_j\}_M(x) \tag{29}$$

for all $x \in N$, where F, G are functions on N and $\hat{F}$, $\hat{G}$ are extensions of F and G to a neighborhood of M. Dirac's formula depends only on F, G, but not on the extensions $\hat{F}$, $\hat{G}$.

When μ is an element of $\mathcal{G}^*$, where $\mathcal{G}$ is a semi–simple Lie algebra, Cushman and Roberts proved that, in suitable coordinates, the transverse structure is polynomial; see [21] and [22].

2.5 Cohomology

Cohomology of Lie algebras was introduced by Chevalley and Eilenberg in [23]. Let $\mathcal{G}$ be a Lie algebra and let ρ be a representation of $\mathcal{G}$ with representation space V. A q-linear skew-symmetric mapping of $\mathcal{G}$ into V will be called a q-dimensional V-cochain. The q-cochains form a space $C^q(\mathcal{G}, V)$. By definition, $C^0(\mathcal{G}, V) = V$.
We define a coboundary operator $\delta = \delta_q : C^q(\mathcal{G}, V) \to C^{q+1}(\mathcal{G}, V)$ by the formula

$$\begin{aligned} (\delta f)(x_0, \ldots, x_q) &= \textstyle\sum_{i=0}^q (-1)^q \rho(x_i) f(x_0, \ldots, \hat{x}_i, \ldots, x_q) + \\ & \textstyle\sum_{i<j} (-1)^{i+j} f([x_i, x_j], x_0, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_q) \ , \end{aligned} \tag{30}$$

where $f \in C^q(\mathcal{G}, V)$ and $x_0, \ldots, x_q \in \mathcal{G}$. As can be easily checked $\delta_{q+1} \circ \delta_q = 0$ so that $\{C^q(\mathcal{G}, V), \delta_q\}$ is an algebraic complex. Define $Z^q(\mathcal{G}, V)$ the space of q-cocycles as the kernel of $\delta : C^q \to C^{q+1}$ and the space $B^q(\mathcal{G}, V)$ of q-coboundaries as the image δC^{q-1}. Since $\delta\delta = 0$ we can define

$$H^q(\mathcal{G}, V) = \frac{Z^q(\mathcal{G}, V)}{B^q(\mathcal{G}, V)} \ . \tag{31}$$

Lichnerowicz [16] considers the following cohomology defined on the tensors of a Poisson manifold. Let (M, π) be a Poisson manifold. If we set $B = C = \pi$ in (5) we get

$$[\pi, [\pi, A]] = 0 \tag{32}$$

for every tensor A. Define a coboundary operator ∂_π which assigns to each p-tensor A, a $(p+1)$-tensor $\partial_\pi A$ given by

$$\partial_\pi A = -[\pi, A] \ . \tag{33}$$

We have $\partial_\pi^2 A = [\pi, [\pi, A]] = 0$ and therefore ∂_π defines a cohomology. An element A is a p-cocycle if $[\pi, A] = 0$. An element B is a p-coboundary if $B = [\pi, C]$, for some $(p-1)$-tensor C. Let

$$Z^n(M, \pi) = \{A \in T_n \,|\, [\pi, A] = 0\} \tag{34}$$

and

$$B^n(M, \pi) = \{B \,|\, B = [\pi, C], \quad C \in T_{n-1}\} \ . \tag{35}$$

The quotient

$$H^n(M, \pi) = \frac{Z^n(M, \pi)}{B^n(M, \pi)} \tag{36}$$

is the nth cohomology group.

Let $\mathcal{G}$ be a Lie algebra and consider the Lie-Poisson manifold $\mathcal{G}^*$. Define a representation ρ of $\mathcal{G}$ with values in $C^\infty(\mathcal{G}^*)$ by

$$\rho(x_i) f = \sum_{j,k} c_{ij}^k \frac{\partial f}{\partial x_j} \tag{37}$$

where x_i denotes coordinates on $\mathcal{G}^*$ and at the same time elements of a basis for $\mathcal{G}$. In other words, $\rho(x_i)f = \{x_i, f\}$, where the bracket is the Lie-Poisson bracket on $\mathcal{G}^*$. We denote the nth cohomology group of $\mathcal{G}$ with respect to this representation by

$$H^n(\mathcal{G}, C^\infty(\mathcal{G}^*)) \ . \tag{38}$$

We have the following result:

Theorem 3

$$H^n(\mathcal{G}^*, \pi) \cong H^n(\mathcal{G}, C^\infty(\mathcal{G}^*)) \ . \tag{39}$$

The proof can be found in [24] or [10].

2.6 Bi–Hamiltonian systems

Proposition 1 *Let (M, π_1), (M, π_2) two Poisson structures on M. The following are equivalent:*
i) $\pi_1 + \pi_2$ is Poisson.
ii) $[\pi_1, \pi_2] = 0$.
iii) $\partial_{\pi_1} \partial_{\pi_2} = -\partial_{\pi_2} \partial_{\pi_1}$.
iv) $\pi_1 \in Z^2(M, \pi_2)$, $\pi_2 \in Z^2(M, \pi_1)$.

Two tensors which satisfy the equivalent conditions are said to form a Poisson pair on M. The corresponding Poisson brackets are called compatible.

Lemma 1 *Suppose π_1 is Poisson and $\pi_2 = L_X \pi_1 = -\partial_{\pi_1} X$ for some vector field X. Then π_1 is compatible with π_2.*

Proof:

$$[\pi_1, \pi_2] = [\pi_1, -[\pi_1, X]] = -\partial_{\pi_1} \partial_{\pi_1} X = 0 \ .$$

■

If π_1 is symplectic, we call the Poisson pair non-degenerate. If we assume a non-degenerate pair we make the following definition: The recursion operator associated with a non-degenerate pair is the $(1,1)$-tensor $\mathcal{R}$ defined by

$$\mathcal{R} = \pi_2 \pi_1^{-1}. \tag{40}$$

A bi-Hamiltonian system is defined by specifying two Hamiltonian functions H_1, H_2 satisfying

$$X = \pi_1 \nabla H_2 = \pi_2 \nabla H_1 \ . \tag{41}$$

We have the following result due to Magri [25]:

Theorem 4 *Suppose that we have a non–degenerate bi-Hamiltonian system on a manifold M, whose first cohomology group is trivial. Then, there exists a hierarchy of mutually commuting functions $H_1, H_2, \ldots$, all in involution with respect to both brackets. If we denote by χ_i the Hamiltonian vector field generated by H_i with respect to the initial bracket π_1 then the χ_i generate mutually commuting bi-Hamiltonian flows, satisfying the Lenard recursion relations*

$$\chi_{i+j} = \pi_i \nabla H_j \ , \tag{42}$$

where $\pi_i = \mathcal{R}^{i-1}\pi_1$ are the higher order Poisson tensors.

For further information on bi-Hamiltonian systems relevant to Toda type systems see [26], [27], [28], [29].

2.7 Master Symmetries

We recall the definition and basic properties of master symmetries following Fuchssteiner [30]. Consider a differential equation on a manifold M defined by a vector field χ. We are mostly interested in the case where χ is a Hamiltonian vector field. A vector field Z is a symmetry of the equation if

$$[Z, \chi] = 0 \ . \tag{43}$$

If Z is time dependent, then a more general condition is

$$\frac{\partial Z}{\partial t} + [Z, \chi] = 0 \ . \tag{44}$$

A vector field Z is called a master symmetry if

$$[[Z, \chi], \chi] = 0 \ , \tag{45}$$

but

$$[Z, \chi] \neq 0 \ . \tag{46}$$

Master symmetries were first introduced by Fokas and Fuchssteiner in [31] in connection with the Benjamin-Ono Equation.
Suppose that we have a bi-Hamiltonian system defined by the Poisson tensors π_1, π_2 and the Hamiltonians H_1, H_2. Assume that π_1 is symplectic. We define the recursion operator $\mathcal{R} = \pi_2\pi_1^{-1}$, the higher flows

$$\chi_i = \mathcal{R}^{i-1}\chi_1 \ , \tag{47}$$

and the higher order Poisson tensors

$$\pi_i = \mathcal{R}^{i-1}\pi_1 \ .$$

For a non-degenerate bi-Hamiltonian system, master symmetries can be generated using a method due to W. Oevel [32].

Theorem 5 *Suppose that X_0 is a conformal symmetry for both π_1, π_2 and H_1, i.e., for some scalars λ, μ, and ν we have*

$$\mathcal{L}_{X_0}\pi_1 = \lambda\pi_1, \qquad \mathcal{L}_{X_0}\pi_2 = \mu\pi_2, \qquad \mathcal{L}_{X_0}H_1 = \nu H_1 \ .$$

Then the vector fields

$$X_i = \mathcal{R}^i X_0$$

are master symmetries and we have
(a)

$$\mathcal{L}_{X_i} H_j = (\nu + (j-1+i)(\mu-\lambda)) H_{i+j}$$

(b)

$$\mathcal{L}_{X_i} \pi_j = (\mu + (j-i-2)(\mu-\lambda)) \pi_{i+j}$$

(c)

$$[X_i, X_j] = (\mu-\lambda)(j-i) X_{i+j} \ .$$

3 A_N TODA LATTICE

3.1 Definition of the system

Equation (2) is the classical, finite, nonperiodic Toda lattice. This system was investigated in [33], [34], [35], [36], [37], [38], [39] and numerous of other papers that are impossible to list here.
This type of Hamiltonian was considered first by Morikazu Toda [39]. The original Toda lattice can be viewed as a discrete version of the Korteweg–de Vries equation. It is called a lattice as in atomic lattice since interatomic interaction was studied. This system also appears in Cosmology. It appears also in the work of Seiberg and Witten on supersymmetric Yang–Mills theories and it has applications in analog computing and numerical computation of eigenvalues. But the Toda lattice is mainly a theoretical mathematical model which is important due to the rich mathematical structure encoded in it.
Hamilton's equations become

$$\begin{aligned} \dot{q}_j &= p_j \\ \dot{p}_j &= e^{q_{j-1}-q_j} - e^{q_j - q_{j+1}} \ . \end{aligned}$$

The system is integrable. One can find a set of independent functions $\{H_1, \dots, H_N\}$ which are constants of motion for Hamilton's equations. To determine the constants of motion, one uses Flaschka's transformation:

$$a_i = \frac{1}{2} e^{\frac{1}{2}(q_i - q_{i+1})} \ , \qquad\qquad b_i = -\frac{1}{2} p_i \ . \tag{48}$$

Then

$$\begin{aligned} \dot{a}_i &= a_i \, (b_{i+1} - b_i) \\ \dot{b}_i &= 2 \, (a_i^2 - a_{i-1}^2) \ . \end{aligned} \tag{49}$$

These equations can be written as a Lax pair $\dot{L} = [B, L]$, where L is the Jacobi matrix

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & \cdots & 0 \\ a_1 & b_2 & a_2 & \cdots & & \vdots \\ 0 & a_2 & b_3 & \ddots & & \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & a_{N-1} \\ 0 & \cdots & & \cdots & a_{N-1} & b_N \end{pmatrix} ,$$

and

$$B = \begin{pmatrix} 0 & a_1 & 0 & \cdots & \cdots & 0 \\ -a_1 & 0 & a_2 & \cdots & & \vdots \\ 0 & -a_2 & 0 & \ddots & & \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & a_{N-1} \\ 0 & \cdots & \cdots & & -a_{N-1} & 0 \end{pmatrix} .$$

This is an example of an isospectral deformation; the entries of L vary over time but the eigenvalues remain constant. It follows that the functions $H_i = \frac{1}{i}\mathrm{tr}\, L^i$ are constants of motion. We note that

$$H_1 = \sum_{i=1}^{N} b_i = -\frac{1}{2}(p_1 + p_2 + \ldots + p_N) ,$$

corresponds to the total momentum and

$$H_2 = H(q_1, \ldots, q_N,\, p_1, \ldots, p_N) = \frac{1}{2}\sum_{i=1}^{N} b_i^2 + \sum_{i=1}^{N-1} a_i^2$$

is the Hamiltonian.
Consider $\mathbf{R}^{2N}$ with coordinates $(q_1, \ldots, q_N, p_1, \ldots, p_N)$, the standard symplectic bracket

$$\{f, g\}_s = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right) , \tag{50}$$

and the mapping $F : \mathbf{R}^{2N} \to \mathbf{R}^{2N-1}$ defined by

$$F : (q_1, \ldots, q_N, p_1, \ldots, p_N) \to (a_1, \ldots, a_{N-1}, b_1, \ldots, b_N) .$$

There exists a bracket on $\mathbf{R}^{2N-1}$ which satisfies

$$\{f, g\} \circ F = \{f \circ F, g \circ F\}_s .$$

It is a bracket which (up to a constant multiple) is given by

$$\begin{array}{lcl} \{a_i, b_i\} & = & -a_i \\ \{a_i, b_{i+1}\} & = & a_i ; \end{array} \tag{51}$$

all other brackets are zero. $H_1 = b_1 + b_2 + \ldots + b_N$ is the only Casimir. The Hamiltonian in this bracket is $H_2 = \frac{1}{2}$ tr L^2. We also have involution of invariants, $\{H_i, H_j\} = 0$. The Lie algebraic interpretation of this bracket can be found in [2]. We denote this bracket by π_1.
The quadratic Toda bracket appears in conjunction with isospectral deformations of Jacobi matrices. First, let λ be an eigenvalue of L with normalized eigenvector v. Standard perturbation theory shows that

$$\nabla\lambda = (2v_1v_2, \ldots, 2v_{N-1}v_N, v_1^2, \ldots, v_N^2)^T := U^\lambda \ ,$$

where $\nabla\lambda$ denotes $(\frac{\partial\lambda}{\partial a_1}, \ldots, \frac{\partial\lambda}{\partial b_N})$. Some manipulations show that U^λ satisfies

$$\pi_2\, U^\lambda = \lambda\, \pi_1\, U^\lambda,$$

where π_1 and π_2 are skew-symmetric matrices. It turns out that π_1 is the matrix of coefficients of the Poisson tensor (51), and π_2, whose coefficients are quadratic functions of the a's and b's, can be used to define a new Poisson tensor. The quadratic Toda bracket appeared in a paper of Adler [40] in 1979. It is a Poisson bracket in which the Hamiltonian vector field generated by H_1 is the same as the Hamiltonian vector field generated by H_2 with respect to the π_1 bracket. The defining relations are

$$\begin{array}{lcl} \{a_i, a_{i+1}\} & = & \frac{1}{2} a_i a_{i+1} \\ \{a_i, b_i\} & = & -a_i b_i \\ \{a_i, b_{i+1}\} & = & a_i b_{i+1} \\ \{b_i, b_{i+1}\} & = & 2\, a_i^2 \ ; \end{array} \tag{52}$$

all other brackets are zero. This bracket has $\det L$ as Casimir and $H_1 = \operatorname{tr} L$ is the Hamiltonian. The eigenvalues of L are still in involution. Furthermore, π_2 is compatible with π_1. We also have

$$\pi_2 \nabla H_l = \pi_1 \nabla H_{l+1} \ . \tag{53}$$

These relations are similar to the Lenard relations for the KdV equation; they are generally called the Lenard relations. Taking $l = 1$ in (53), we conclude that the Toda lattice is bi–Hamiltonian. In fact, using results from [15], we can prove that the Toda lattice is multi–Hamiltonian:

$$\pi_2\, \nabla H_1 = \pi_1\, \nabla H_2 = \pi_0\, \nabla H_3 = \pi_{-1}\, \nabla H_4 = \ldots \tag{54}$$

Finally, we remark that further manipulations with the Lenard relations for the infinite Toda lattice, followed by setting all but finitely many a_i, b_i equal to zero, yield another Poisson bracket, π_3, which is cubic in the coordinates; see [41]. The defining relations for π_3 are

$$\begin{array}{lcl} \{a_i, a_{i+1}\} & = & a_i a_{i+1} b_{i+1} \\ \{a_i, b_i\} & = & -a_i b_i^2 - a_i^3 \\ \{a_i, b_{i+1} & = & a_i b_{i+1}^2 + a_i^3 \\ \{a_i, b_{i+2}\} & = & a_i a_{i+1}^2 \\ \{a_{i+1}, b_i\} & = & -a_i^2 a_{i+1} \\ \{b_i, b_{i+1}\} & = & 2\, a_i^2\, (b_i + b_{i+1}) \ ; \end{array} \tag{55}$$

all other brackets are zero. The bracket π_3 is compatible with both π_1 and π_2 and the eigenvalues of L are still in involution. The Casimir for this bracket is $\operatorname{tr} L^{-1}$.

The multi-Hamiltonian structure of the Toda lattice is well-known. The results are usually presented either in the natural (q, p) coordinates or in the more convenient Flaschka coordinates (a, b). In the former case the hierarchy of higher invariants are generated by the use of a recursion operator [42], [43]. In the later case one uses master symmetries as in [9], [10]. We have to point out that chronologically every result obtained so far was done first in Flaschka coordinates (a, b) and then transferred through the inverse of Flaschka's transformation to the original (q, p) coordinates. This is to be expected since it is always easier to work with sums of polynomials than with sums of exponentials.
The sequence of Poisson tensors can be extended to form an infinite hierarchy. In order to produce the hierarchy of Poisson tensors one uses master symmetries. The first three Poisson brackets are precisely the linear, quadratic and cubic brackets we mentioned above. If a system is bi-Hamiltonian and one of the brackets is symplectic, one can find a recursion operator by inverting the symplectic tensor. The recursion operator is then applied to the initial symplectic bracket to produce an infinite sequence. However, in the case of Toda lattice (in Flaschka variables (a, b)) both operators are non-invertible and therefore this method fails. The absence of a recursion operator for the finite Toda lattice is also mentioned in Morosi and Tondo [44] where a Nijenhuis tensor for the infinite Toda lattice is calculated. Recursion operators were introduced by Olver [45].

3.2 Multi–Hamiltonian structure

In the case of Toda equations, the master symmetries map invariant functions to other invariant functions. Hamiltonian vector fields are also preserved. New Poisson brackets are generated by using Lie derivatives in the direction of these vector fields and they satisfy interesting deformation relations. We give a summary of the results:

- There exists a sequence of invariants

$$H_1, H_2, H_3, \dots,$$

  where $H_i = \frac{1}{i}\mathrm{Tr}\, L^i$,

- a corresponding sequence of Hamiltonian vector fields

$$\chi_1, \quad \chi_2, \quad \chi_3, \dots,$$

  where $\chi_i = \chi_{H_i}$,

- a hierarchy of Poisson tensors

$$\pi_1, \pi_2, \pi_3, \dots,$$

  where π_i is polynomial, homogeneous, of degree i.

- Finally, one can determine a sequence of master symmetries

$$X_1, X_2, X_3, \dots,$$

  which are used to create the hierarchies through Lie derivatives.

We quote the results from refs. [9], [10].

Theorem 6
i) $\pi_j, \;\; j \geq 1$ *are all Poisson.*

ii) The functions H_i, $i \geq 1$ *are in involution with respect to all of the* π_j.

iii) $X_i(H_j) = (i+j)H_{i+j}$, $i \geq -1$, $j \geq 1$.

iv) $L_{X_i}\pi_j = (j-i-2)\pi_{i+j}$, $i \geq -1$, $j \geq 1$.

v) $[X_i, \; X_j] = (j-i)X_{i+j}$, $i \geq 0$, $j \geq 0$.

vi) $\pi_j \nabla H_i = \pi_{j-1} \nabla H_{i+1}$, *where* π_j *denotes the Poisson matrix of the tensor* π_j.

To define the vector fields X_n one considers expressions of the form

$$\dot{L} = [B, L] + L^{n+1} \; . \tag{56}$$

This equation is similar to a Lax equation, but in this case the eigenvalues satisfy $\dot{\lambda} = \lambda^{n+1}$ instead of $\dot{\lambda} = 0$.
We give an outline of the construction of the vector fields X_n. We define X_{-1} to be

$$\nabla \; H_1 = \nabla \; \text{Tr} \; L = \sum_{i=1}^{N} \frac{\partial}{\partial b_i} \; , \tag{57}$$

and X_0 to be the Euler vector field

$$X_0 = \sum_{i=1}^{N-1} a_i \frac{\partial}{\partial a_i} + \sum_{i=1}^{N} b_i \frac{\partial}{\partial b_i} \; . \tag{58}$$

We want X_1 to satisfy

$$X_1(\text{Tr} \; L^n) = n \text{Tr} \; L^{n+1} \; . \tag{59}$$

One way to find such a vector field is by considering the equation

$$\dot{L} = [B, L] + L^2 \; . \tag{60}$$

Note that the left hand side of this equation is a tridiagonal matrix while the right hand side is pentadiagonal. We look for B as a tridiagonal matrix

$$B = \begin{pmatrix} \gamma_1 & \beta_1 & 0 & \cdots & \cdots \\ \alpha_1 & \gamma_2 & \beta_2 & \cdots & \cdots \\ 0 & \alpha_2 & \gamma_3 & \beta_3 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix} \; . \tag{61}$$

We want to choose the α_i, β_i and γ_i so that the right hand side of equation (60) becomes tridiagonal. One simple solution is $\alpha_n = -(n+1)a_n$, $\beta_n = (n+1)a_n$, $\gamma_n = 0$. The vector field X_1 is defined by the right hand side of (60):

$$X_1 = \sum_{n=1}^{N-1} \dot{a}_n \frac{\partial}{\partial a_n} + \sum_{n=1}^{N} \dot{b}_n \frac{\partial}{\partial b_n} \; , \tag{62}$$

where

$$\dot{a}_n = -na_nb_n + (n+2)a_nb_{n+1} \tag{63}$$

$$\dot{b}_n = (2n+3)a_n^2 + (1-2n)a_{n-1}^2 + b_n^2 \ . \tag{64}$$

To construct the vector field X_2 we consider the equation

$$\dot{L} = [B, L] + L^3 \ .$$

The calculations are similar to those for X_1. The matrix B is now pentadiagonal and the system of equations slightly more complicated. The result is a vector field

$$X_2 = \sum_{n=1}^{N-1} \dot{a}_n \frac{\partial}{\partial a_n} + \sum_{n=1}^{N} \dot{b}_n \frac{\partial}{\partial b_n}$$

where

$$\dot{a}_n = (2-n)a_{n-1}^2a_n + (1-n)a_nb_n^2 + a_nb_nb_{n+1} + (n+1)a_na_{n+1}^2 + (n+1)a_nb_{n+1}^2 + a_n^3 + \sigma_na_n(b_{n+1} - b_n)$$

$$\dot{b}_n = 2\sigma_na_n^2 - 2\sigma_{n-1}a_{n-1}^2 + (2n+2)a_n^2b_n + (2n+1)a_n^2b_{n+1} + + (3-2n)a_{n-1}^2b_{n-1} + (4-2n)a_{n-1}^2b_n + b_n^3 \ ,$$

with

$$\sigma_n = \sum_{i=1}^{n-1} b_i \ ,$$

and $\sigma_1 = 0$.
We continue the sequence of master symmetries for $n \geq 3$ by

$$[X_1, X_{n-1}] = (n-2)X_n \ . \tag{65}$$

3.3 Properties of X_n and π_n.

It is well known that π_1, π_2, π_3 satisfy Lenard relations

$$\pi_n \nabla H_l = \pi_{n-1} \nabla H_{l+1} \ , \qquad n = 2, 3 \qquad \forall l \ . \tag{66}$$

We want to show that these relations hold for all values of n. We denote the Hamiltonian vector field of H_l with respect to the nth bracket by χ_l^n. In other words,

$$\chi_l^n = [\pi_n, H_l] \ . \tag{67}$$

We prove the Lenard relations in an equivalent form.

Proposition 2 $\chi_l^{n+1} = \chi_{l+1}^n$.

Proof: To prove this we need the identity

$$[X_1, \chi_l^n] = (n-3)\chi_l^{n+1} + (l+1)\chi_{l+1}^n \tag{68}$$

which follows easily from $X_1(H_l) = (l+1)H_{l+1}$ and (67). Therefore,

$$\begin{aligned}
(n-3)\chi_l^{n+1} &= [X_1, \chi_l^n] - (l+1)\chi_{l+1}^n \\
&= [X_1, \chi_{l+1}^{n-1} - (l+1)\chi_{l+1}^n \\
&= (n-4)\chi_{l+1}^n + (l+2)\chi_{l+2}^{n-1} - (l+1)\chi_{l+1}^n \\
&= (n-4)\chi_{l+1}^n + (l+2)\chi_{l+1}^n - (l+1)\chi_{l+1}^n \\
&= (n-3)\chi_{l+1}^n \qquad . \qquad \blacksquare
\end{aligned} \tag{69}$$

Using the Lenard relations we can show that the functions H_n are in involution with respect to all of the brackets π_n.

Proposition 3 *$\{H_i, H_j\}_n = 0$, where $\{\ ,\ \}_n$ is the bracket corresponding to π_n.*

Proof: First we consider the Lie-Poisson Toda bracket. We have

$$\{H_1, H_j\} = 0 \qquad \forall j\ , \tag{70}$$

since H_1 is a Casimir for π_1. Suppose that $\{H_{i-1}, H_j\} = 0\ \ \forall\ j$.

$$\begin{aligned}
i\{H_i, H_j\} &= \{X_1(H_{i-1}), H_j\} \\
&= -[\chi_j^1, [X_1, H_{i-1}]] \\
&= [X_1, \{H_{i-1}, H_j\}] + [H_{i-1}, (j+1)\chi_{j+1}^1] \\
&= (j+1)\{H_{i-1}, H_{j+1}\} \\
&= 0\ .
\end{aligned} \tag{71}$$

Now we use induction on n. Suppose

$$\{H_i, H_j\}_n = 0 \qquad \forall\ i,\ j\ . \tag{72}$$

$$\begin{aligned}
\{H_i, H_j\}_{n+1} &= \chi_i^{n+1}(H_j) \\
&= \chi_{i+1}^n(H_j) \\
&= \{H_{i+1}, H_j\}_n \\
&= 0\ .
\end{aligned} \tag{73}$$

Of course, one can prove the involution of integrals without using master symmetries. We present the classical proof:
In this proof the symbol π_n stands for the bundle map π_n^* defined by (18). We first prove the involution of constants in the Lie-Poisson Toda bracket. We use the basic Lenard relation

$$\pi_2\, dH_1 = \pi_1\, dH_2\ .$$

Since H_1 is a Casimir in the linear bracket, we have $\pi_1 dH_1 = 0$. We calculate

$$\begin{aligned}
\{H_i, H_j\}_1 &= < dH_i,\ \pi_1 dH_j > \\
&= - < dH_j,\ \pi_1 dH_i > \\
&= - < dH_j,\ \pi_2 dH_{i-1} > \\
&= < dH_{i-1},\ \pi_2 dH_j > \\
&= < dH_{i-1},\ \pi_1 dH_{j+1} > \\
&= \{H_1,\ H_{j+i-1}\}_1 = 0\ .
\end{aligned}$$

It is also easy to show involution in the second quadratic bracket:

$$\begin{aligned}\{H_i, H_j\}_2 &= < dH_i,\ \pi_2 dH_j > \\ &= < dH_i,\ \pi_1 dH_{j+1} > \\ &= \{H_i,\ H_{j+1}\}_1 = 0\ .\end{aligned}$$

The general result follows from the Lenard relations $\pi_n dH_j = \pi_{n-1} dH_{j+1}$ and induction:

$$\begin{aligned}\{H_i, H_j\}_n &= < dH_i, \pi_n dH_j > \\ &= < dH_i, \pi_{n-1} dH_{j+1} > \\ &= \{H_i, H_{j+1}\}_{n-1} = 0\ .\end{aligned}$$

■

It is straightforward to verify that the mapping

$$f(a_1, \ldots, a_{N-1}, b_1, \ldots, b_N) = (a_1, \ldots, a_{N-1}, 1 + b_1, \ldots, 1 + b_N) \tag{74}$$

is a Poisson map between π_2 and $\pi_1 + \pi_2$. Since f is a diffeomorphism, we have the isomorphism

$$\pi_2 \cong \pi_1 + \pi_2\ . \tag{75}$$

In other words, the tensor π_2 encodes sufficient information for both the linear and quadratic Toda brackets. An easy induction generalizes this result: i.e.,

Proposition 4

$$\pi_n \cong \sum_{j=0}^{n-1} \binom{n-1}{j} \pi_{n-j}\ . \tag{76}$$

The function $\mathrm{tr}\ L^{2-n}$, which is well-defined on the open set $\det\ L \neq 0$, is a Casimir for π_n, for $n \geq 3$. The proof uses the Lenard type relation

$$\pi_n \nabla \lambda = \lambda \pi_{n-1} \nabla \lambda \tag{77}$$

satisfied by the eigenvalues of L. To prove the last equation, one uses the relation

$$\pi_n \sum \lambda_k^{l-1} \nabla \lambda_k = \pi_{n-1} \sum \lambda_k^l \nabla \lambda_k\ . \tag{78}$$

But

$$\sum \lambda_k^{l-1} (\pi_k \nabla \lambda_k - \lambda_k \pi_{n-1} \nabla \lambda_k) = 0\ , \tag{79}$$

for $l = 1, 2, \ldots, N+1$, has only the trivial solution because the (Vandermonde) coefficient determinant is nonzero.

Proposition 5 *For $n > 2$, $\mathrm{tr}\, L^{2-n}$ is a Casimir for π_n on the open dense set $\det L \neq 0$.*

Proof: For $n = 3$,

$$\begin{aligned}\pi_3 \nabla \mathrm{tr}\, L^{-1} &= \pi_3 \textstyle\sum_k -\frac{1}{\lambda_k^2} \nabla \lambda_k \\ &= \textstyle\sum_k -\frac{1}{\lambda_k^2} \lambda_k \pi_2 \nabla \lambda_k \\ &= -\textstyle\sum_k \pi_1 \nabla \lambda_k \\ &= -\pi_1 \nabla\ \mathrm{tr}\, L = \chi_1^1 = 0\ .\end{aligned} \tag{80}$$

For $n > 3$ the induction step is as follows:

$$\begin{array}{rcl} \pi_n \nabla \mathrm{tr}\, L^{2-n} & = & \pi_n \nabla \sum_k \frac{1}{\lambda_k^{n-2}} \\ & = & \sum_k (2-n) 1 \lambda_k^{n-1} \pi_n \nabla \lambda_k \\ & = & \sum_k (2-n) \frac{1}{\lambda_k^{n-1}} \lambda_k \pi_{n-1} \nabla \lambda_k \\ & = & \frac{n-2}{n-3} \pi_{n-1} \nabla \mathrm{tr}\, L^{3-n} \\ & = & 0 \ . \end{array} \tag{81}$$

3.4 The Faybusovich–Gekhtman approach

In [46] Faybusovich and Gekhtman find another method of generating the multi–Hamiltonian structure for the Toda lattice. Their method is important and will certainly have applications to other integrable systems, both finite and infinite dimensional, solvable by the inverse spectral transform. Their work shows that the Hamiltonian formalism is built into the spectral theory.

In the case of Toda lattice, the key ingredient is the Moser map which takes the (a, b) phase space of tridiagonal Jacobi matrices to a new space of variables (λ_i, r_i) where λ_i is an eigenvalue of the Jacobi matrix and r_i is the residue of rational functions that appear in the solution of Toda equations. The Poisson brackets of Theorem 6 project onto some rational brackets in the space of Weyl functions and in particular, the Lie–Poisson bracket π_1 corresponds to the Atiyah–Hitchin bracket [47]. We briefly describe the construction: Moser in [48] introduced the resolvent

$$R(\lambda) = (\lambda I - L)^{-1} \ ,$$

and defined the Weyl function

$$f(\lambda) = (R(\lambda)e_1,\ e_1) \ ,$$

where $e_1 = (1, 0, \ldots, 0)$.

The function $f(\lambda)$ has a simple pole at $\lambda = \lambda_i$ and positive residue at λ_i equal to r_i:

$$f(\lambda) = \sum_{i=1}^{n} \frac{r_i}{\lambda - \lambda_i} \ .$$

The variables (a, b) may be expressed as rational functions of λ_i and r_i using a continued fraction expansion of $f(\lambda)$ which dates back to Stieltjes. Since the computation of the continued fraction from the partial fraction expansion is a rational process the solution is expressed as a rational function of the variables $(\lambda_i,\ r_i)$. The idea of Faybusovich and Gehtman is to construct a sequence of Poisson brackets on the space (λ_i, r_i) whose image under the inverse spectral transform are the brackets π_i defined in Theorem 6. The Lie–Poisson bracket π_1 corresponds to the Atiyah–Hitchin bracket on Weyl functions which in coordinate free form is written as

$$\{f(\lambda),\ f(\mu)\} = \frac{(f(\lambda) - f(\mu))^2}{\lambda - \mu} \ .$$

A rational function of the form $\frac{q(\lambda)}{p(\lambda)}$ is determined uniquely by the dinstinct eigenvalues of $p(\lambda)$, $\lambda_1, \ldots, \lambda_n$ and values of q at these roots. The residue r_i is equal to $\frac{q(\lambda_i)}{p'(\lambda_i)}$ and therefore we may choose

$$\lambda_1, \dots, \lambda_n, q(\lambda_1), \dots, q(\lambda_n)$$

as global coordinates on the space of rational functions (of the form $\frac{q}{p}$ with p having simple roots and q, p coprime). We have to remark that the image of the Moser map is a much larger set.
The kth Poisson bracket is defined by

$$\begin{array}{rcl} \{\lambda_i,\ q(\lambda_i)\} & = & -\lambda_i^k q(\lambda_i) \\ \{q(\lambda_i),\ q(\lambda_j)\} & = & \{\lambda_i,\ \lambda_j\} = 0 \ . \end{array}$$

Let us denote this bracket by w_k.
On the other hand in [9], page 108, there is a definition of vector fields on the space of eigenvalues of the Jacobi matrix which are projections of the master symmetries X_i. They are defined by

$$e_i = \sum_{j=1}^{N} \lambda_j^{i+1} \frac{\partial}{\partial \lambda_j} \ .$$

One verifies easily that these vector fields satisfy the usual Virasoro type relation

$$[e_i,\ e_j] = (j-i) e_{i+j} \ .$$

If we denote by F the function which sends the Jacobi matrix to its eigenvalues then

$$dF(X_1) = e_1$$

$$dF(X_2) = e_2 \ .$$

Therefore, it follows by induction that

$$dF(X_i) = e_i \ .$$

Faybusovich and Gekhtman used the brackets w_i and the vector fields e_j to obtain the analogue of Theorem 6 in the space of rational functions. The relations obtained correspond to the relations of Theorem 6 under the inverse of the Moser map.
The explicit formulas for for the brackets w_k can be deduced easily from the formulas in [46]. They are

$$\begin{array}{rcl} \{r_i,\ r_j\}_k & = & \frac{\lambda_i^k + \lambda_j^k}{\lambda_i - \lambda_j} r_i r_j \\ \{r_i,\ \lambda_i\}_k & = & \lambda_i r_i \\ \{\lambda_i,\ \lambda_j\}_k & = & 0 \ . \end{array}$$

In a recent paper [49] Vaninsky has also explicit formulas in $(\lambda_i,\ r_i)$ coordinates for the initial bracket w_1.

3.5 A Theorem of Petalidou

Finally, we mention an interesting result of Petalidou [50]. She proves the following Theorem: Suppose that $(M, \Lambda_0, \Lambda_1)$ is a bi–Hamiltonian manifold of odd dimension and let p be a point in M of corank 1. If there exists locally an infinitesimal automorphism Z_0 of Λ_0 which

is transverse to the symplectic leaf through p and a vector field Z_1 which depends on a parameter t such that

$$[\Lambda_1, Z_1] + [Z_1, \frac{\partial}{\partial t}] \wedge Z_1 = 0 ,$$

and

$$[\Lambda_0, Z_1] + [\Lambda_1, Z_0] + [Z_1, \frac{\partial}{\partial t}] \wedge Z_0 = 0 ,$$

then one can find a symplectic realization of both Λ_0 and Λ_1 by a pair of symplectic brackets $\hat{\Lambda}_0$, $\hat{\Lambda}_1$ given by

$$\hat{\Lambda}_i = \Lambda_i + Z_i \wedge \frac{\partial}{\partial t} \qquad i = 1, 2 \quad .$$

This result applies in the case of the Toda lattice by taking $Z_0 = X_{-1}$ and $Z_1 = X_0$. She obtains symplectic realizations of π_1 and π_2. Furthermore, she constructs symplectic realizations of the whole sequence

$$\pi_1, \pi_2, \pi_3, \ldots$$

of Theorem 6. The corresponding symplectic sequence is given by

$$\hat{\pi}_k = \pi_k + X_{k-2} \wedge \frac{\partial}{\partial t} \ .$$

The tensors $\hat{\pi}_k$ may be generated by a recursion operator since the initial tensor, which is a multiple of $\hat{\Lambda}_0$, is invertible.

3.6 A recursive process of Kosmann–Schwarzbach and Magri

In [51] Y. Kosmann–Schwarzbach and F. Magri consider the relationship between Lax and bi–Hamiltonian formulations of integrable systems. They introduce an equation, called the Lax-Nijenhuis equation, relating the Lax matrix with the bi–Hamiltonian pair and they show that every operator that satisfies that equation satisfies also the Lenard recursion relations. They derive the multi–Hamiltonian structure of the Toda lattice by defining a matrix M and a vector λ_0 which arise by manipulating the Lax–Nijenhuis equation. They show that

$$\pi_2 = M\pi_1 + X \otimes \lambda_0 ,$$

where X is the Hamiltonian vector field χ_2. In the next step of the recursive process they show that

$$\pi_3 = M\pi_2 + X \otimes \lambda_1 ,$$

where $\lambda_1 = M\lambda_0$. In general,

$$\pi_{i+1} = M\pi_i + X \otimes M^{(i-1)}\lambda_0 \ .$$

4 LIE GROUP SYMMETRIES OF THE TODA LATTICE

Sophus Lie introduced his theory of continuous groups in order to study symmetry properties of differential equations. His approach allowed a unification of existing methods for solving ordinary differential equations as well as classifications of symmetry groups of partial and ordinary differential equations. A symmetry group of a system of differential equations is a

Lie group acting on the space of independent and dependent variables in such a way that solutions are mapped into other solutions. Knowing the symmetry group allows one to determine some special types of solutions invariant under a subgroup of the full symmetry group, and in some cases one can solve the equations completely. Lie's methods have been developed into powerful tools for examining differential equations through group analysis. In many cases, symmetry groups are the only known means for finding concrete solutions to complicated equations. The method applies of course to the case of Hamiltonian or Lagrangian systems, both autonomous and time dependent. Recently, the immense amount of computations needed for determining symmetry groups of concrete systems has been greatly reduced by the implementation of computer algebra packages for symmetry analysis of differential equations. The symmetry approach to solving differential equations can be found, for example, in the books of Olver [52], Bluman and Kumei [53], Ovsiannikov [54] and Ibragimov [55].

Some properties of master symmetries are clear: They preserve constants of motion, Hamiltonian vector fields and they generate a hierarchy of Poisson brackets. We are interested in the following problem : Can one find a symmetry group of the system whose infinitesimal generator is a given master symmetry? In the case of Toda equations the answer is negative. However, in this section we find a sequence consisting of time dependent evolution vector fields whose time independent part is a master symmetry. Each master symmetry X_n can be written in the form $Y_n + tZ_n$ where Y_n is a time dependent symmetry and Z_n is a time independent Hamiltonian symmetry (i.e. a Hamiltonian vector field).

In other words, we find an infinite sequence of evolution vector fields that are symmetries of equations (49). We do not know if every symmetry of Toda equations is included in this sequence.

We begin by writing equations (49) in the form

$$\Gamma_j = \dot{a}_j - a_j b_{j+1} + a_j b_j = 0 \quad \Delta_j = \dot{b}_j - 2a_j^2 + 2a_{j-1}^2 = 0 \ .$$

We look for symmetries of Toda equations. i.e. vector fields of the form

$$\mathbf{v} = \tau \frac{\partial}{\partial t} + \sum_{j=1}^{N-1} \phi_j \frac{\partial}{\partial a_j} + \sum_{j=1}^{N} \psi_j \frac{\partial}{\partial b_j}$$

that generate the symmetry group of the Toda equations. The first prolongation of $\mathbf{v}$ is

$$\mathrm{pr}^{(1)}\mathbf{v} = \mathbf{v} + \sum_{j=1}^{N-1} f_j \frac{\partial}{\partial \dot{a}_j} + \sum_{j=1}^{N} g_j \frac{\partial}{\partial \dot{b}_j} \ ,$$

where

$$\begin{aligned} f_j &= \dot{\phi}_j - \dot{\tau}\dot{a}_j \\ g_j &= \dot{\psi}_j - \dot{\tau}\dot{b}_j \ . \end{aligned}$$

The infinitesimal condition for a group to be a symmetry of the system is

$$\mathrm{pr}^{(1)}(\Gamma_j) = 0 \qquad \mathrm{pr}^{(1)}(\Delta_j) = 0 \ .$$

Therefore we obtain the equations

$$\dot{\phi}_j - \dot{\tau} a_j (b_{j+1} - b_j) + \phi_j (b_j - b_{j+1}) + a_j \psi_j - a_j \psi_{j+1} = 0 \ , \tag{82}$$

$$\dot{\psi}_j - 2\dot{\tau}(a_j^2 - a_{j-1}^2) - 4a_j\phi_j + 4a_{j-1}\phi_{j-1} = 0 \ . \tag{83}$$

We first give some obvious solutions :

i) $\tau = 0$, $\phi_j = 0$, $\psi_j = 1$. This is the vector field X_{-1}.

ii) $\tau = -1$, $\phi_j = 0$, $\psi_j = 0$. The resulting vector field is the time translation $-\frac{\partial}{\partial t}$ whose evolutionary representative is

$$\sum_{j=1}^{N-1} \dot{a}_j \frac{\partial}{\partial a_j} + \sum_{j=1}^{N} \dot{b}_j \frac{\partial}{\partial b_j} \ .$$

This is the Hamiltonian vector field χ_{H_2}. It generates a Hamiltonian symmetry group.

iii) $\tau = -1$, $\phi_j = a_j$, $\psi_j = b_j$. Then

$$\mathbf{v} = -\frac{\partial}{\partial t} + \sum_{j=1}^{N-1} a_j \frac{\partial}{\partial a_j} + \sum_{j=1}^{N} b_j \frac{\partial}{\partial b_j} = -\frac{\partial}{\partial t} + X_0 \ .$$

This vector field generates the same symmetry as the evolutionary vector field

$$X_0 + t\chi_{H_2} \ .$$

We next look for some non obvious solutions. The vector field X_1 is not a symmetry, so we add a term which depends on time. We try

$$\phi_j = -ja_jb_j + (j+2)a_jb_{j+1} + t(a_ja_{j+1}^2 + a_jb_{j+1}^2 - a_{j-1}^2a_j - a_jb_j^2)$$

$$\psi_j = (2j+3)a_j^2 + (1-2j)a_{j-1}^2 + b_j^2 + t(2a_j^2b_{j+1} + 2a_j^2 - 2a_{j-1}^2a_j - 2a_{j-1}^2b_j) \ ,$$

and $\tau = 0$.

A tedious but straightforward calculation shows that ϕ_j, ψ_j satisfy (82) and (83). It is also straightforward to check that the vector field

$$\sum \phi_j \frac{\partial}{\partial a_j} + \sum \psi_j \frac{\partial}{\partial b_j}$$

is precisely equal to $X_1 + t\chi_{H_3}$. The pattern suggests that $X_n + t\chi_{H_{n+2}}$ is a symmetry of Toda equations.

Theorem 7 *The vector fields $X_n + t\chi_{n+2}$ are symmetries of Toda equations for $n \geq -1$.*

Proof : Note that $\chi_{H_1} = 0$ because H_1 is a Casimir for the Lie-Poisson bracket. We use the formula

$$[X_n, \chi_l] = (l-1)\chi_{n+l} \ . \tag{84}$$

In particular, for $l = 2$, we have $[X_n, \chi_2] = \chi_{n+2}$.

Since the Toda flow is Hamiltonian, generated by χ_2, to show that $Y_n = X_n + t\chi_{n+2}$ are symmetries of Toda equations we must verify the equation

$$\frac{\partial Y_n}{\partial t} + [\chi_2, Y_n] = 0 \ . \tag{85}$$

But

$$\begin{aligned} \tfrac{\partial Y_n}{\partial t} + [\chi_2,\ Y_n] &= \tfrac{\partial Y_n}{\partial t} + [\chi_2,\ X_n + t\chi_{n+2}] \\ &= \chi_{n+2} - [X_n,\ \chi_2] \\ &= \chi_{n+2} - \chi_{n+2} = 0 \ . \end{aligned} \tag{86}$$

$\blacksquare$

5 THE TODA LATTICE IN NATURAL COORDINATES

In this section we define the positive and negative Toda hierarchies for the Toda lattice in (q, p) variables. We follow reference [15].

5.1 The Das–Okubo–Fernandes approach

Another approach, which explains the relations of Theorem 6 is adopted in Das and Okubo [42], and Fernandes [43]. In principle, their method is general and may work for other finite dimensional systems as well. This approach was also used in [56] by da Costa and Marle in the case of the Relativistic Toda lattice. The procedure is the following: One defines a second Poisson bracket in the space of canonical variables $(q_1, \dots, q_N, p_1, \dots, p_N)$. This gives rise to a recursion operator. The presence of a conformal symmetry as defined in Oevel [32] allows one, by using the recursion operator, to generate an infinite sequence of master symmetries. These, in turn, project to the space of the new variables (a, b) to produce a sequence of master symmetries in the reduced space.
Let $\hat{J}_1$ be the symplectic bracket (50) with Poisson matrix

$$\hat{J}_1 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} ,$$

where I is the $N \times N$ identity matrix. We use $J_1 = 4\hat{J}_1$. With this convention the bracket J_1 is mapped precisely onto the bracket π_1 under the Flaschka transformation (48). We define $\hat{J}_2$ to be the tensor

$$\hat{J}_2 = \begin{pmatrix} A & B \\ -B & C \end{pmatrix} ,$$

where A is the skew-symmetric matrix defined by $a_{ij} = 1 = -a_{ji}$ for $i < j$, B is the diagonal matrix $(-p_1, -p_2, \dots, -p_N)$ and C is the skew-symmetric matrix whose non-zero terms are $c_{i,i+1} = -c_{i+1,i} = e^{q_i - q_{i+1}}$ for $i = 1, 2, \dots, N-1$. We define $J_2 = 2\hat{J}_2$. With this convention the bracket J_2 is mapped precisely onto the bracket π_2 under the Flaschka transformation. It is easy to see that we have a bi-Hamiltonian pair. We define

$$h_1 = -2(p_1 + p_2 + \ldots + p_N) ,$$

and h_2 to be the Hamiltonian:

$$h_2 = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + \sum_{i=1}^{N-1} e^{q_i - q_{i+1}} .$$

Under Flaschka's transformation (48), h_1 is mapped onto $4(b_1 + b_2 + \ldots + b_N) = 4 \ \mathrm{tr} L = 4H_1$ and h_2 is mapped onto $2 \ \mathrm{tr} L^2 = 4H_2$. Using the relationship

$$\pi_2 \nabla \ H_1 = \pi_1 \nabla \ H_2 ,$$

which follows from Proposition 2, we obtain, after multiplication by 4, the following pair:

$$J_1 \nabla \ h_2 = J_2 \nabla \ h_1 .$$

We define the recursion operator as follows:

$$\mathcal{R} = J_2 J_1^{-1} .$$

The matrix form of $\mathcal{R}$ is quite simple:

$$\mathcal{R} = \frac{1}{2}\begin{pmatrix} B & -A \\ C & B \end{pmatrix} \ . \tag{87}$$

This operator raises degrees and we therefore call it the positive Toda operator. In (q, p) coordinates, the symbol χ_i is a shorthand for χ_{h_i}. It is generated as usual by

$$\chi_i = \mathcal{R}^{i-1}\chi_1 \ .$$

In a similar fashion we obtain the higher order Poisson tensors

$$J_i = \mathcal{R}^{i-1}J_1 \ .$$

We finally define the conformal symmetry

$$Z_0 = \sum_{i=1}^{N}(N-2i+1)\frac{\partial}{\partial q_i} + \sum_{i=1}^{N} p_i \frac{\partial}{\partial p_i} \ .$$

It is straightforward to verify that

$$\mathcal{L}_{Z_0} J_1 = -J_1 \ ,$$

$$\mathcal{L}_{Z_0} J_2 = 0 \ .$$

In fact, Z_0 is Hamiltonian in the J_2 bracket with Hamiltonian function $\frac{1}{2}\sum_{i=1}^{N} q_i$; see [43]. This observation will be generalized in 5.3.
In addition,

$$Z_0(h_1) = h_1$$

$$Z_0(h_2) = 2h_2 \ .$$

Consequently, Z_0 is a conformal symmetry for J_1, J_2 and h_1. The constants appearing in Oevel's Theorem are $\lambda = -1$, $\mu = 0$ and $\nu = 1$. Therefore, we end–up with the following deformation relations:

$$[Z_i, h_j] = (i+j)h_{i+j}$$

$$L_{Z_i} J_j = (j-i-2)J_{i+j}$$

$$[Z_i, Z_j] = (j-i)Z_{i+j} \ .$$

Switching to Flaschka coordinates, we obtain relations iii)- v) of Theorem 6.

5.2 The negative Toda hierarchy

To define the negative Toda hierarchy we use the inverse of the positive recursion operator $\mathcal{R}$. We define

$$\mathcal{N} = \mathcal{R}^{-1} = J_1 J_2^{-1} \ .$$

Obviously we can use the same conformal symmetry $Z_0 = K_0$ and take $\lambda = 0$, $\mu = -1$ and $\nu = 2$. In other words the role of λ and μ is reversed. We define the vector fields

$$K_i = \mathcal{N}^i K_0 = \mathcal{N}^i Z_0 \qquad i = 1, 2, \ldots$$

which are master symmetries. We use the convention $Y_{-i} = K_i$ for $i = 0, 1, 2, \ldots$. For example, $Y_{-1} = K_1 = \mathcal{N} Z_0 = -2\sum_{i=1}^{N} \frac{\partial}{\partial p_i}$. This vector field, in (a, b) coordinates, is given by

$$X_{-1} = \nabla\ H_1 = \nabla\ \mathrm{tr}\ L = \sum_{i=1}^{N} \frac{\partial}{\partial b_i} \ .$$

This is precisely the same vector field (57). In that section, X_{-1} was constructed through a different method. Similarly, the vector field Z_0 corresponds to the Euler vector field (58):

$$X_0 = \sum_{i=1}^{N-1} a_i \frac{\partial}{\partial a_i} + \sum_{i=1}^{N} b_i \frac{\partial}{\partial b_i} \ .$$

Note: We use the symbol Y_i for a vector field in (q, p) coordinates and X_i for the same vector field in (a, b) coordinates. Similarly, we denote by J_i a Poisson tensor in (p, q) coordinates and π_i the corresponding Poisson tensor in (a, b) coordinates. The index i ranges over all integers.

We now calculate, using Oevel's Theorem:

$$[Y_{-i}, Y_{-j}] = [K_i, K_j] = (\mu - \lambda)(j - i)K_{i+j} = (-1)(j - i)K_{i+j} = (i - j)Y_{-(i+j)} \ .$$

Letting $m = -i$ and $n = -j$ we obtain the relationship

$$[Y_m, Y_n] = (n - m)Y_{m+n} \ , \tag{88}$$

for all m, n negative. The same relation holds in Flaschka coordinates. In other words

$$[X_m, X_n] = (n - m)X_{m+n} \qquad \forall\, m, n \in \mathbf{Z}^- \ .$$

This last relation may be modified to hold for any two arbitrary integers m, n. We suppose, without loss of generality, that $j > i$ and consider the bracket of two master symmetries $K_i = Y_{-i}$ and $Z_j = Y_j$, one in the negative hierarchy and the second in the positive hierarchy. i.e.

$$K_i = \mathcal{N}^i Z_0 = R^{-i} Z_0 \ ,$$

and

$$Z_j = \mathcal{R}^j Z_0 \ .$$

We proceed as in the proof of Oevel's Theorem (see [43]): First we note that

$$\mathcal{L}_{Z_0}\mathcal{R} = (\mathcal{L}_{Z_0} J_2)\, J_1^{-1} - J_2 J_1^{-1} \mathcal{L}_{Z_0} J_1 J_1^{-1} = (\mu - \lambda)\mathcal{R} \ .$$

On the other hand

$$\mathcal{L}_{Z_0}\mathcal{N} = \mathcal{L}_{Z_0}\left(J_1 J_2^{-1}\right) = (\lambda-\mu)\mathcal{N} \ .$$

Finally,

$$\begin{aligned}
[Y_{-i},\ Y_j] &= [K_i, Z_j] = [\mathcal{N}^i Z_0, \mathcal{R}^j Z_0] \\
&= \mathcal{N}^i \mathcal{L}_{Z_0}\left(\mathcal{R}^j\right) Z_0 - \mathcal{R}^j \mathcal{L}_{Z_0}\left(\mathcal{N}^i\right) Z_0 \\
&= \mathcal{N}^i j(\mu-\lambda)\mathcal{R}^j Z_0 - \mathcal{R}^j i(\lambda-\mu)\mathcal{N}^i Z_0 \\
&= j(\mu-\lambda)\mathcal{R}^{j-i} Z_0 - i(\lambda-\mu)\mathcal{R}^{j-i} Z_0 \\
&= (i+j)(\mu-\lambda)\mathcal{R}^{j-i} Z_0 \\
&= (i+j)(\mu-\lambda) Y_{j-i} \ .
\end{aligned}$$

In the case of Toda lattice $\mu = 0$ and $\lambda = -1$, therefore

$$[Y_{-i}, Y_j] = (i+j)Y_{j-i} \ .$$

We deduce that (88) holds for any integer value of the index.
We define $W_i = J_{3-i}$. This is necessary since the conclusions of Oevel's Theorem assume that the index begins at $i = 1$ and is positive. We compute

$$\mathcal{L}_{Y_{-i}} J_{-j} = \mathcal{L}_{K_i} W_{j+3} = (\mu + (j+3-2-i)(\mu-\lambda))W_{i+j+3} = (i-j-2)W_{i+j+3} = (i-j-2)J_{-(i+j)} \ .$$

Letting $m = -i$ and $n = -j$ we obtain

$$L_{Y_m} J_n = (n-m-2)J_{n+m} \ ,$$

for n, m negative integers. Switching to Flaschka coordinates we deduce that the relation iv) of Theorem 6 holds also for negative values of the index. In other words

$$L_{X_i}\pi_j = (j-i-2)\pi_{i+j}, \quad i \le 0, \quad j \le 0 \ .$$

Again, a straightforward modification of the proof of Oevel's Theorem shows that the last relationship holds for any integer value of m, n. We have shown that conclusions iv) and v) of Theorem 6 hold for integer values of the index. In fact, it is not difficult to demonstrate all the other parts of Theorem 6.

Theorem 8 *The conclusions of Theorem 6 hold for any integer value of the index.*

Proof:
We need to prove parts *i)*, *ii)*, *iii)* and *vi)* of the Theorem.

i) The fact that J_n are Poisson for $n \in \mathbf{Z}$ follows from properties of the recursion operator. The similar result in (a, b) coordinates follows easily from properties of the Schouten bracket, and the fact that J_n and π_n are $F-$related. We have $\pi_n = F_* J_n$, therefore

$$[\pi_n, \pi_n] = [F_*(J_n), F_*(J_n)] = F_*[J_n, J_n] = F_*(0) = 0 \ .$$

The vanishing of the Schouten bracket is equivalent to the Poisson property.
iii) The case where i and j are both of the same sign was already proved. We next note that $X_n(\lambda) = \lambda^{n+1}$ if λ is an eigenvalue of L. This follows from equation (56) which is used

to define the vector fields X_n for $n \geq 0$. We would like to extend the formula $X_n(\lambda) = \lambda^{n+1}$ for $n < 0$. Since $X_{-1}(\lambda) = 1$ we consider X_{-2}. We look at the equation

$$[X_{-2}, X_n] = (n+2)X_{n-2} \ .$$

We act on λ with both sides of the equation and let $X_{-2}(\lambda) = f(\lambda)$. We obtain the equation

$$(n+1)\lambda f(\lambda) - f'(\lambda)\lambda^2 = (n+2) \ .$$

This is a linear first order ordinary differential equation with general solution

$$f(\lambda) = \frac{1}{\lambda} + c\lambda^{n+1} \ .$$

Since n is arbitrary, we obtain $f(\lambda) = \frac{1}{\lambda}$. In order to calculate $X_{-3}(\lambda)$ we use

$$X_{-3} = -[X_{-1}, X_{-2}] \ .$$

We obtain

$$X_{-3}(\lambda) = X_{-2}X_{-1}(\lambda) - X_{-1}X_{-2}(\lambda) = \ - X_{-1}(\frac{1}{\lambda}) = \frac{1}{\lambda^2} \ .$$

The result follows by induction.
Finally we calculate

$$X_i(H_j) = \frac{1}{j}X_i\left(\sum \lambda_k^j\right) = \frac{1}{j}\left(\sum X_i\lambda_k^j\right) = \sum \lambda_k^{j-1}X_i(\lambda_k) = \sum \lambda_k^{j-1}\lambda_k^{i+1} = \sum \lambda_k^{i+j} = (i+j)H_{i+j} \ .$$

vi) First we note that $\pi_j \ \nabla H_i \ = \ \pi_{j-1} \ \nabla H_{i+1}$, holds for i, j of the same sign. More generally, in the positive (or the negative) hierarchy we have the Lenard relations (77) for the eigenvalues, i.e.

$$\pi_j \nabla \lambda_i = \lambda_i \pi_{j-1} \nabla \lambda_i \ . \tag{89}$$

Assume now that $i < 0$, $j > 0$. The calculation is straightforward:

$$\begin{aligned}
\pi_j \nabla \tfrac{1}{i} \textstyle\sum \lambda_k^i &= \textstyle\sum \lambda_k^{i-1} \pi_j \nabla \lambda_i \\
&= \textstyle\sum \lambda_i^l \pi_{j-1} \nabla \lambda_k \\
&= \pi_{j-1} \nabla \tfrac{1}{i+1} \textstyle\sum \lambda_k^{i+1} \ .
\end{aligned}$$

Therefore,

$$\pi_j \nabla H_i = \pi_{j-1} \nabla H_{i+1} \ . \tag{90}$$

In the case $i > 0$ and $j < 0$ we use exactly the same calculation but use (89) for the negative hierarchy.
ii) It is clearly enough to show the involution of the eigenvalues of L since H_i are functions of the eigenvalues. We prove involution of eigenvalues by using the Lenard relations (90). We give the proof for the case of the bracket π_j with $j > 0$ but if $j < 0$ the proof is identical. First we show that the eigenvalues are in involution with respect to the bracket π_1. Let λ and μ be two distinct eigenvalues and let U, V be the gradients of λ and μ respectively. We use the notation $\{\ ,\ \}$ to denote the bracket π_1 and $\langle\ ,\ \rangle$ the standard inner product. The Lenard relations (89) translate into $\pi_2 U = \lambda \pi_1 U$ and $\pi_2 V = \mu \pi_1 V$. Therefore,

$$\begin{aligned}
\{\lambda,\mu\} &= \langle \pi_1 U, V\rangle = \frac{1}{\lambda}\langle \pi_2 U, V\rangle \\
&= -\frac{1}{\lambda}\langle U, \pi_2 V\rangle = -\frac{1}{\lambda}\langle U, \mu\pi_1 V\rangle \\
&= -\frac{\mu}{\lambda}\langle U, \pi_1 V\rangle = \frac{\mu}{\lambda}\langle \pi_1 U, V\rangle \\
&= \frac{\mu}{\lambda}\{\lambda,\mu\}\ .
\end{aligned}$$

Therefore, $\{\lambda,\mu\} = 0$. To show the involution with respect to all brackets π_j, and in view of part *iv)* of Theorem 6, it is enough to show the following: Let f_1, f_2 be two functions in involution with respect to the Poisson bracket π, let X be a vector field such that $X(f_i) = f_i^2$ for $i = 1, 2$. Define a Poisson bracket w by $w = \mathcal{L}_X\pi$. Then the functions f_1, f_2 remain in involution with respect to the bracket w. The proof follows trivially if we write $w = \mathcal{L}_X\pi$ in Poisson form:

$$\{f_1, f_2\}_w = X\{f_1, f_2\}_\pi - \{f_1, X(f_2)\}_\pi - \{X(f_1), f_2\}_\pi\ .$$

■

Remark: We should point out that

$$H_n = \frac{1}{n}\,\mathrm{tr}L^n\ ,$$

makes sense for $n \neq 0$ but it is undefined for $n = 0$. The reader should interpret the formulas involving H_0 as a degenerate case, i.e. $H_0 = \frac{\mathrm{tr}L^0}{0} = \frac{N}{0} = \infty$. Therefore the result of $X_{-n}(H_n) = N$ where N is the size of L. It makes sense to define

$$X_m(H_0) = \lim_{n\to 0}\frac{1}{n}X_m\left(\mathrm{tr}\,L^n\right)\ .$$

For example, $X_{-1}(H_0)$ is calculated by $X_{-1}\left(\frac{1}{n}\,\mathrm{tr}\,L^n\right) = \mathrm{tr}\,L^{n-1}$. Taking the limit as $n \to 0$ gives $X_{-1}(H_0) = \mathrm{tr}\,L^{-1} = -H_{-1}$ which is the correct answer.

5.3 Master integrals and master symmetries

In this section we prove some further results and give some specific examples.
In subsection 5.1 we noticed that Z_0 is Hamiltonian with respect to the J_2 bracket with Hamiltonian function $f = \frac{1}{2}\sum_{i=1}^N q_i$. This observation is due to Fernandes [43]. This type of function is called a master integral. It is not a constant of motion, but its derivative is. We generalize the result as follows:

Theorem 9 *The master symmetry Y_n, $n \in \mathbf{Z}$ is the Hamiltonian vector field of f with respect to the J_{n+2} bracket.*

Proof: We will prove the result for the positive hierarchy $Z_n = Y_n$ but the proof for $Y_{-n} = K_n$ is similar. As a first step we show that

$$Z_n(f) = 0 \quad \forall\ \ n \geq 0\ .$$

We recall that

$$Z_0=\sum_{i=1}^N (N-2i+1)\frac{\partial}{\partial q_i}+\sum_{i=1}^N p_i\frac{\partial}{\partial p_i}\ .$$

Since

$$\sum_{i=1}^N (N+1-2i)=0\ ,$$

we obtain

$$Z_0(f)=\frac{1}{2}Z_0(\sum_{i=1}^N q_i)=\frac{1}{2}(\sum_{i=1}^N Z_0(q_i))=0\ .$$

By examining the form (87) of the recursion operator $\mathcal{R}$ we deduce easily that the q_i component of Z_1 is

$$Z_1(q_i)=-\frac{1}{2}\left[(N-2i+1)p_i+\sum_{j>i}p_j-\sum_{j<i}p_j\right]\ .$$

In other words, the vector

$$(Z_1(q_1),\ldots,Z_1(q_N))$$

is the product AP where

$$A=-\frac{1}{2}\begin{pmatrix} Z_0(q_1) & 1 & 1 & \cdots & \cdots & 1\\ -1 & Z_0(q_2) & 1 & \cdots & \cdots & \vdots\\ -1 & -1 & Z_0(q_3) & \ddots & & \\ \vdots & & \ddots & \ddots & & \vdots\\ \vdots & & & \ddots & \ddots & 1\\ -1 & \cdots & & \cdots & -1 & Z_0(q_N)\end{pmatrix},$$

and P is the column vector $(p_1,p_2,\ldots,p_N)^t$. Note that $\sum_{i=1}^N a_{ij}=0$ and $\sum_{j=1}^N a_{ij}=-Z_0(q_i)$. Therefore,

$$Z_1(f)=\frac{1}{2}(Z_1(q_1)+\ldots+Z_1(q_N))=\frac{1}{2}\sum_{i,j}a_{ij}p_j=\frac{1}{2}\left(\sum_j\left(\sum_i a_{ij}\right)p_j\right)=0\ .$$

In the same fashion one proves that $Z_2(f)=0$.
For $n>2$, we proceed by induction.

$$Z_n=\frac{1}{n-2}\left[Z_1,Z_{n-1}\right]\ .$$

Therefore,

$$Z_n(f)=\frac{1}{n-2}\left[Z_1,Z_{n-1}\right](f)=\frac{1}{n-2}\left(Z_1Z_{n-1}f-Z_{n-1}Z_1f\right)=0\ ,$$

by the induction hypothesis.
To complete the proof of the Theorem, it is enough to show

$$Z_n=[J_{n+2},f]\ ,$$

where [,] denotes the Schouten bracket.
First we note that

$$[[J_{n+1}, f], Z_1] + [[f, Z_1], J_{n+1}] + [[Z_1, J_{n+1}], f] = 0$$

due to the super Jacobi identity for the Schouten bracket. Since

$$[Z_1, f] = Z_1(f) = 0 \ ,$$

the middle term in the last identity is zero. We obtain

$$[Z_1, [J_{n+1}, f]] = [[Z_1, J_{n+1}], f] \ .$$

Finally, we calculate using induction:

$$\begin{aligned} Z_n &= \frac{1}{n-2}[Z_1, Z_{n-1}] \\ &= \frac{1}{n-2}[Z_1, [J_{n+1}, f]] \\ &= \frac{1}{n-2}[[Z_1, J_{n+1}], f] \\ &= \frac{1}{n-2}\left([(n-2)J_{n+2}, f]\right) \\ &= [J_{n+2}, f] \ . \end{aligned}$$

■

The result of the Theorem is striking. It shows that the master symmetries are determined once the Poisson hierarchy is constructed. Of course one requires knowledge of the function f. The function f may be constructed by using Noether's Theorem: The symmetries of the Toda lattice in (q, p) coordinates have been constructed in [57], at least for two degrees of freedom. The Lie algebra for the potential of the Toda lattice with N degrees of freedom is five dimensional with generators

$$\begin{aligned} X_1 &= \tfrac{\partial}{\partial t} \\ X_2 &= t\tfrac{\partial}{\partial t} - 2\sum_{i=2}^{N}(i-1)\frac{\partial}{\partial q_i} \\ X_3 &= \left(\sum_{i=1}^{N} q_i\right)\sum_{i=1}^{N}\frac{\partial}{\partial q_i} \\ X_4 &= \sum_{i=1}^{N}\frac{\partial}{\partial q_i} \\ X_5 &= t\sum_{i=1}^{N}\frac{\partial}{\partial q_i} \ . \end{aligned} \tag{91}$$

The non–zero bracket relations satisfied by the generators are

$$\begin{aligned} [X_1, X_2] &= X_1 \\ [X_1, X_5] &= X_4 \\ [X_2, X_3] &= -2X_4 \\ [X_2, X_5] &= X_5 \\ [X_3, X_4] &= -2X_4 \\ [X_3, X_5] &= -2X_5 \ . \end{aligned}$$

This Lie algebra L is solvable with $L^{(1)} = [L, L] = \{X_1, X_4, X_5\}$, $L^{(2)} = \{X_4\}$ and $L^{(3)} = \{0\}$. We examine the symmetry X_5.
A corresponding time dependent integral produced from Noether's Theorem is

$$I = \frac{1}{2}\sum_{i=1}^{N} q_i - \frac{1}{2}t\sum_{i=1}^{N} p_i = f + \frac{1}{4}th_1 \ .$$

Motivated by the results of [58], [59], it makes sense to consider the time independent part of I which is precisely the function f. It is an interesting question whether this procedure works for other integrable systems as well. We also remark that the integrals are also determined from the knowledge of the Poisson brackets and the function f. For example, it follows easily from Theorem 9 that

$$h_{i+1} = \frac{1}{i+1}\{h_i, f\}_3 \ ,$$

where $\{\ ,\ \}_3$ denotes the cubic Toda bracket.

5.4 Noether Symmetries

We recall that Noether's Theorem states that for a first order Lagrangian, the action integral $\int_{t_1}^{t_2} L dt$ is invariant under the infinitesimal transformation generated by the differential operator, known as a Noether symmetry,

$$X = T\frac{\partial}{\partial t} + \sum_{i=1}^{N} Q_i \frac{\partial}{\partial q_i} \tag{92}$$

if there exists a function F, known as a gauge term, such that

$$\dot{F} = T\frac{\partial L}{\partial t} + \sum_{i=1}^{N} Q_i \frac{\partial L}{\partial q_i} + \sum_{i=1}^{N}\left(\dot{Q}_i - \dot{q}_i\dot{T}\right)\frac{\partial L}{\partial \dot{q}_i} + \dot{T}L \ . \tag{93}$$

When the corresponding Euler–Lagrange equation is taken into account, equation (93) can be manipulated to yield the first integral

$$I = F - \left[TL + \sum_{i=1}^{N}\left(Q_i - \dot{q}_i T\right)\frac{\partial L}{\partial \dot{q}_i}\right] \ . \tag{94}$$

Thus to every Noether symmetry there is an associated first integral. Consider the Lagrangian to be of the form

$$L = \frac{1}{2}\sum_{i=1}^{N} \dot{q}_i^{\,2} - V(q_1, q_2, \ldots, q_N) \ . \tag{95}$$

We summarize the results of [60] in the following Theorem:

Theorem 10 *Let* $\mathbf{X}$ *be the* $N \times 1$ *vector with entries* Q_i*,* $\mathbf{x}$ *the vector with entries* q_i *and* $\mathbf{b}$ *the vector with entries* $b_i(t)$*. Let* $\mathbf{A}$ *be an* $N \times N$ *skew-symmetric matrix with constant entries. Finally we denote by* $\mathbf{I_N}$ *the* $N \times N$ *identity matrix. If* X*, given by (92), is a Noether symmetry then the infinitesimals must be of the form*

$$\begin{aligned} T &= T(t) \\ \mathbf{X} &= \left(\mathbf{A} + \frac{1}{2}\frac{dT}{dt}\mathbf{I_N}\right)\mathbf{x} + \mathbf{b} \end{aligned} \tag{96}$$

and the gauge term is restricted to

$$F = \frac{1}{4}\frac{d^2T}{dt^2}\sum_{i=1}^{N} q_i^2 + \sum_{i=1}^{N} \frac{db_i(t)}{dt} q_i + d(t). \tag{97}$$

The associated first integral I is equal to

$$F + TH - \sum_{i=1}^{N} Q_i p_i \ ,$$

where H is the Hamiltonian.
By examining the form (91) of the generators for the Toda lattice we conclude, using Theorem 10, that only X_1, X_4 and X_5 are Noether symmetries. The corresponding integrals provided by Noether's Theorem are the Hamiltonian $H = h_2$, the total momentum $h_1 = p_1 + \ldots + p_N$ and $f + th_1$.
In order to obtain more integrals we consider generalized Noether symmetries. That is, the infinitesimals in (92) do not just depend on $t, q_1, \ldots, q_N$ but also on $\dot{q}_1, \ldots, \dot{q}_N$. For Lagrangians of the form (95) for one, two and three degrees of freedom, all the possible point Noether symmetries are classified in [60]. The following results are from [61].
In the case of generalized symmetries, we can take without loss of generality $T = 0$. Hence, we consider operators of the form

$$G = \sum_{i=1}^{N} \eta_i \frac{\partial}{\partial q_i} \tag{98}$$

where the infinitesimals of (92) and (98) are related by

$$\eta_i = Q_i - \dot{q}_i T \ .$$

Using (95) and (98), Noether's condition (93) becomes

$$f_t + \sum_{i=1}^{N} \dot{q}_i f_{q_i} + \sum_{i=1}^{N} \ddot{q}_i f_{\dot{q}_i} = \sum_{i=1}^{N} \eta_i V_{q_i} + \sum_{i=1}^{N} \dot{q}_i \left(\eta_{it} + \sum_{j=1}^{N} \dot{q}_j \eta_{iq_j} + \sum_{j=1}^{N} \ddot{q}_j \eta_{i\dot{q}_j} \right) \ . \tag{99}$$

We consider equation (99) in the case of the Toda lattice with two degrees of freedom. By assuming various forms of the η_i (i.e. linear, quadratic or arbitrary) we can solve this equation and produce the following integrals one of which (I_3) is new:

$$I_1 = p_1 + p_2 \ , \qquad I_2 = (p_1 - p_2)^2 + 4e^{q_1 - q_2} \ ,$$

$$I_3 = \frac{p_1 - p_2 + \sqrt{I_2}}{p_1 - p_2 - \sqrt{I_2}} \exp\left(\sqrt{I_2} \frac{q_1 + q_2}{p_1 + p_2} \right) \ .$$

Note that $H = \frac{1}{4}\left(I_1^2 + I_2\right)$ and that the function $G = \frac{q_1+q_2}{p_1+p_2}$ which appears in the exponent of I_3 satisfies $\{G, H\} = 1$.
The existence of the integral I_3 shows that the two degrees of freedom Toda lattice is super–integrable with three integrals of motion $\{I_1, I_2, I_3\}$. A Hamiltonian system with N degrees of freedom is called super-integrable if it possesses $2N - 1$ independent integrals of motion. Of course these integrals cannot be all in involution. Based on this computation for 2 degrees of freedom we make the following conjecture:
Conjecture: The Toda lattice is super–integrable.

5.5 Rational Poisson brackets

The rational brackets in (q, p) coordinates are given by complicated expressions that are quite hard to write in explicit form. When projected in the space of (a, b) variables they give rational brackets whose numerator is polynomial and the denominator is the determinant of the Jacobi matrix. We give examples of these brackets and master symmetries for $N = 3$. For example, the tensor J_0 is a homogeneous rational bracket of degree 0. It is defined by

$$J_0 = \mathcal{N} J_1 = J_1 J_2^{-1} J_1 \ .$$

In the case of three particles the corresponding bracket π_0 is given as follows: First define the skew-symmetric matrix A by

$$\begin{aligned}
a_{12} &= -\frac{1}{2}a_1 a_2 (b_3 + b_1 - b_2) \\
a_{13} &= a_1(a_2^2 - b_2 b_3) \\
a_{14} &= -a_1(a_2^2 - b_1 b_3) \\
a_{15} &= a_1 a_2^2 \\
a_{23} &= -a_1^2 a_2 \\
a_{24} &= a_2(a_1^2 - b_1 b_3) \\
a_{25} &= -a_2(a_1^2 - b_1 b_2) \\
a_{34} &= -2a_1^2 b_3 \\
a_{35} &= 0 \\
a_{45} &= -2a_2^2 b_1 \ .
\end{aligned}$$

The matrix of the tensor π_0 is defined by

$$\pi_0 = \frac{1}{\det L} A \tag{100}$$

where $\det L = b_1 b_2 b_3 - a_2^2 b_1 - a_1^2 b_3$. This formula defines a Poisson bracket with one single Casimir $H_2 = \frac{1}{2}\mathrm{tr}\, L^2$. The bracket is defined on the open dense set $\det L \neq 0$. Taking $H_3 = \frac{1}{3}\mathrm{tr}\, L^3$ as the Hamiltonian we have another bi–Hamiltonian formulation of the system:

$$\pi_1 \, dH_2 = \pi_0 \, dH_3 \ .$$

In fact we have infinite pairs of such formulations since

$$\pi_2 \, dH_1 = \pi_1 \, dH_2 = \pi_0 \, dH_3 = \pi_{-1} \, dH_4 = \ldots$$

The explicit formulas for the vector fields X_1 and X_2 are given in 3.1 therefore we give an example for the vector field X_{-2}. In the case $N = 3$ it is given by

$$X_{-2} = \frac{1}{\det L} \left(\sum_{i=1}^{2} r_i \frac{\partial}{\partial a_i} + \sum_{i=1}^{3} s_i \frac{\partial}{\partial b_i} \right)$$

where

$$\begin{array}{rcl} r_1 & = & \frac{1}{2}a_1(b_1 - b_2 - 2b_3) \\ r_2 & = & \frac{1}{2}a_2(b_3 - 2b_1 - b_2) \\ s_1 & = & b_2b_3 - a_1^2 - a_2^2 \\ s_2 & = & b_1b_3 + a_1^2 + a_2^2 \\ s_3 & = & b_1b_2 - a_1^2 - a_2^2 \ . \end{array}$$

Finally, we consider the Casimirs of these new Poisson brackets.

Theorem 11 *The Casimir of π_n in the open dense set $\det L \neq 0$ is Tr L^{2-n} for all $n \neq 2$. The Casimir of π_2 is det L.*

Proof:
For $n \geq 1$ the result was proved in Proposition 5. Therefore, we only have to show that the Casimir of π_{-m} is tr L^{m+2} $(m \geq 0)$. This follows from (90) and the fact that $H_1 = \operatorname{Tr} L$ is the Casimir for the Lie-Poisson bracket π_1:

$$0 = \pi_1 \nabla H_1 = \pi_0 \nabla H_2 = \pi_{-1} \nabla H_3 = \dots \ .$$

6 GENERALIZED TODA SYSTEMS ASSOCIATED WITH SIMPLE LIE GROUPS

In this section we consider mechanical systems which generalize the finite, nonperiodic Toda lattice. These systems correspond to Dynkin diagrams. They are special cases of (1) where the spectrum corresponds to a system of simple roots for a simple Lie algebra. It is well known that irreducible root systems classify simple Lie groups. So, in this generalization for each simple Lie algebra there exists a mechanical system of Toda type.

The generalization is obtained from the following simple observation: In terms of the natural basis q_i of weights, the simple roots of A_{n-1} are

$$q_1 - q_2, \, q_2 - q_3, \dots, q_{n-1} - q_n \ . \tag{101}$$

On the other hand, the potential for the Toda lattice is of the form

$$e^{q_1 - q_2} + e^{q_2 - q_3} + \dots + e^{q_{n-1} - q_n} \ . \tag{102}$$

We note that the angle between $q_{i-1} - q_i$ and $q_i - q_{i+1}$ is $\frac{2\pi}{3}$ and the lengths of $q_i - q_{i+1}$ are all equal. The Toda lattice corresponds to a Dynkin diagram of type A_{n-1}.

More generally, we consider potentials of the form

$$U = c_1 \, e^{f_1(q)} + \dots + c_l \, e^{f_l(q)} \tag{103}$$

where $c_1, \dots, c_l$ are constants, $f_i(q)$ is linear and l is the rank of the simple Lie algebra. For each Dynkin diagram we construct a Hamiltonian system of Toda type. These systems are

interesting not only because they are integrable, but also for their fundamental importance in the theory of semisimple Lie groups. For example Kostant in [2] shows that the integration of these systems and the theory of the finite dimensional representations of semisimple Lie groups are equivalent.

For reference, we give a complete list of the Hamiltonians for each simple Lie algebra.

$\underline{A_{n-1}}$

$$H = \frac{1}{2}\sum_1^n p_j^2 + e^{q_1-q_2} + \cdots + e^{q_{n-1}-q_n}$$

$\underline{B_n}$

$$H = \frac{1}{2}\sum_1^n p_j^2 + e^{q_1-q_2} + \cdots + e^{q_{n-1}-q_n} + e^{q_n}$$

$\underline{C_n}$

$$H = \frac{1}{2}\sum_1^n p_j^2 + e^{q_1-q_2} + \cdots + e^{q_{n-1}-q_n} + e^{2q_n}$$

$\underline{D_n}$

$$H = \frac{1}{2}\sum_1^n p_j^2 + e^{q_1-q_2} + \cdots + e^{q_{n-1}-q_n} + e^{q_{n-1}+q_n}$$

$\underline{G_2}$

$$H = \frac{1}{2}\sum_1^3 p_j^2 + e^{q_1-q_2} + e^{-2q_1+q_2+q_3}$$

$\underline{F_4}$

$$H = \frac{1}{2}\sum_1^4 p_j^2 + e^{q_1-q_2} + e^{q_2-q_3} + e^{q_3} + e^{\frac{1}{2}(q_4-q_1-q_2-q_3)}$$

$\underline{E_6}$

$$H = \frac{1}{2}\sum_1^8 p_j^2 + \sum_1^4 e^{q_j-q_{j+1}} + e^{-(q_1+q_2)} + e^{\frac{1}{2}(-q_1+q_2+\ldots+q_7-q_8)}$$

$\underline{E_7}$

$$H = \frac{1}{2}\sum_1^8 p_j^2 + \sum_1^5 e^{q_j-q_{j+1}} + e^{-(q_1+q_2)} + e^{\frac{1}{2}(-q_1+q_2+\ldots+q_7-q_8)}$$

$\underline{E_8}$

$$H = \frac{1}{2}\sum_1^8 p_j^2 + \sum_1^6 e^{q_j-q_{j+1}} + e^{-(q_1+q_2)} + e^{\frac{1}{2}(-q_1+q_2+\ldots+q_7-q_8)}$$

We should note that the Hamiltonians in the list are not unique. For example, the A_2 Hamiltonian is

$$H = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + \frac{1}{2}p_3^2 + e^{q_1-q_2} + e^{q_2-q_3} \ . \tag{104}$$

An equivalent system is

$$H(Q_i, P_i) = \frac{1}{2}P_1^2 + \frac{1}{2}P_2^2 + e^{\sqrt{\frac{2}{3}}(\sqrt{3}Q_1+Q_2)} + e^{-2\sqrt{\frac{2}{3}}Q_2} \ . \tag{105}$$

The second Hamiltonian is obtained from the first by using the transformation

$$Q_1 = \frac{\sqrt{2}}{4}(q_1 + q_2 - 2q_3) \tag{106}$$

$$Q_2 = \frac{\sqrt{6}}{4}(q_2 - q_1) \tag{107}$$

$$P_1 = \frac{2}{\sqrt{2}}(p_1 + p_2) \tag{108}$$

$$P_2 = \frac{2}{\sqrt{6}}(p_2 - p_1) \ . \tag{109}$$

Another example is the following two systems, both corresponding to a Lie algebra of type D_4:

$$\sum_{i=1}^{4} \frac{p_i^2}{2} + e^{q_1} + e^{q_2} + e^{q_3} + e^{\frac{1}{2}(q_4-q_1-q_2-q_3)}$$

$$\sum_{i=1}^{4} \frac{p_i^2}{2} + e^{q_1-q_2} + e^{q_2-q_3} + e^{q_3-q_4} + e^{q_3+q_4} \ .$$

Finally, let us recall the definition of exponents for a semi-simple group G. An excellent reference is the book by Collingwood and McGovern [62]. Let G be a connected, complex, simple Lie Group G. We form the de Rham cohomology groups $H^i(G, \mathbf{C})$ and the corresponding Poincaré polynomial of G:

$$p_G(t) = \sum d_i t^i \ ,$$

where d_i =dim $H^i(G, \mathbf{C})$. A Theorem of Hopf shows that the cohomology algebra is a finite product of l spheres of odd dimension where l is the rank of G.. This Theorem implies that

$$p_G(t) = \prod_{i=1}^{l}(1 + t^{2e_i+1}) \ .$$

The positive integers $\{e_1, e_2, ..., e_l\}$ are called the *exponents* of G. One can also extract the exponents from the root space decomposition of G. The connection with the invariant polynomials is the following: Let $H_1, H_2, ..., H_l$ be independent, homogeneous, invariant polynomials of degrees $m_1, m_2, .., m_l$. Then $e_i = m_i - 1$. The exponents of a simple Lie group are given in the following list:

$\underline{A_{n-1}}$

$$1, 2, 3, ..., n-1$$

$\underline{B_n}$, $\underline{C_n}$

$$1, 3, 5, ..., 2n-1$$

$\underline{D_n}$

$$1, 3, 5, ..., 2n-3, n-1$$

$\underline{G_2}$

$$1, 5$$

$\underline{F_4}$

$$1, 5, 7, 11$$

$\underline{E_6}$

$$1, 4, 5, 7, 8, 11$$

$\underline{E_7}$

$$1, 5, 7, 9, 11, 13, 17$$

$\underline{E_8}$

$$1, 7, 11, 13, 17, 19, 23, 29$$

7 B_n TODA SYSTEMS

7.1 A rational bracket for a central extension of B_n-Toda

In this subsection we show that the B_n Toda system is bi–Hamiltonian by considering a central extension of the the corresponding Lie algebra in analogy with $gl(n, \mathbf{C})$ which is a central extension of $sl(n, \mathbf{C})$ in the case of A_n Toda.

Another way to describe these generalized Toda systems, is to give a Lax pair representation in each case. It can be shown that the equation $\dot{L} = [B, L]$ is equivalent to the equations of motion generated by the Hamiltonian $H_2 = \frac{1}{2}\mathrm{tr}\, L^2$ on the orbit through L of the coadjoint action of B_- (lower triangular group) on the dual of its Lie algebra, $\mathcal{B}_-^*$. The space $\mathcal{B}_-^*$ can be identified with the set of symmetric matrices. This situation, which corresponds to $sl(n, \mathbf{C}) = A_{n-1}$ can be generalized to other semisimple Lie algebras. We use notation and definitions from Humphreys [63].

Let $\mathcal{G}$ be a semisimple Lie algebra, Φ a root system for $\mathcal{G}$, $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ the simple roots, h a Cartan subalgebra and $\mathcal{G}_\alpha$ the root space of α. We denote by x_α a generator of $\mathcal{G}_\alpha$. Define

$$\mathcal{B}_- = h \oplus \sum_{\alpha<0} \mathcal{G}_\alpha \ .$$

There is an automorphism σ of $\mathcal{G}$, of order 2, satisfying $\sigma(x_\alpha) = x_{-\alpha}$ and $\sigma(x_{-\alpha}) = x_\alpha$. Let $\mathcal{K} = \{x \in \mathcal{G} \,|\, \sigma(x) = -x\}$. Then we have a direct sum decomposition $\mathcal{G} = \mathcal{B}_- \oplus \mathcal{K}$. The Toda flow is a coadjoint flow on $\mathcal{B}_-^*$ and the coadjoint invariant functions on $\mathcal{G}^*$, when restricted to $\mathcal{B}_-^*$ are still in involution. The Jacobi elements are of the form

$$L = \sum_{i=1}^{l} b_i h_i + \sum_{i=1}^{l} a_i \left(x_{\alpha_i} + x_{-\alpha_i} \right) .$$

We define

$$B = \sum_{i=1}^{l} a_i \left(x_{\alpha_i} - x_{-\alpha_i} \right) .$$

The generalized Toda flow takes the Lax pair form:

$$\dot{L} = [B, L] .$$

The B_n Toda systems were shown to be Bi-Hamiltonian. The second bracket can be found in [10]. It turned out to be a rational bracket and it was obtained by using Dirac's constrained bracket formula (29). The idea is to use the inclusion of B_n into A_{2n} and to restrict the hierarchy of brackets from A_{2n} to B_n via Dirac's bracket. Straightforward restriction does not work. We briefly describe the procedure in the case of B_2.

The Jacobi matrices for A_4 and B_2 are given by

$$L_{A_4} = \begin{pmatrix} b_1 & a_1 & 0 & 0 & 0 \\ a_1 & b_2 & a_2 & 0 & 0 \\ 0 & a_2 & b_3 & a_3 & 0 \\ 0 & 0 & a_3 & b_4 & a_4 \\ 0 & 0 & 0 & a_4 & b_5 \end{pmatrix} , \tag{110}$$

and

$$L_{B_2} = \begin{pmatrix} b_1 & a_1 & 0 & 0 & 0 \\ a_1 & b_2 & a_2 & 0 & 0 \\ 0 & a_2 & b_3 & -a_2 & 0 \\ 0 & 0 & -a_2 & 2b_3 - b_2 & -a_1 \\ 0 & 0 & 0 & -a_1 & 2b_3 - b_1 \end{pmatrix} . \tag{111}$$

Note that L_{A_4} lies in $\mathrm{gl}(4, \mathbf{C})$ instead of $\mathrm{sl}(4, \mathbf{C})$. Therefore we have added an additional variable in L_{B_2}. We define

$$\begin{aligned} p_1 &= a_1 + a_4 \\ p_2 &= a_2 + a_3 \\ p_3 &= b_1 + b_5 - 2b_3 \\ p_4 &= b_2 + b_4 - 2b_3 \ . \end{aligned}$$

It is clear that we obtain B_2 from A_4 by setting $p_i = 0$ for $i = 1, 2, 3, 4$. We calculate the matrix $P = \{p_i, p_j\}$. The bracket used is the quadratic Toda (52) on A_4.

$$
\begin{aligned}
\{p_1, p_2\} &= \frac{1}{2}(a_1a_2 - a_3a_4) \\
\{p_1, p_3\} &= a_4b_5 - a_1b_1 \\
\{p_1, p_4\} &= a_1b_2 - a_4b_4 \\
\{p_2, p_3\} &= 2(a_3b_3 - 2a_2b_3) \\
\{p_2, p_4\} &= a_3b_4 + 2a_3b_3 - 2a_2b_3 - a_2b_2 \\
\{p_3, p_4\} &= -2a_4^2 - 4a_3^2 + 4a_2^2 + 2a_1^2 \ .
\end{aligned}
$$

If we evaluate at a point in B_2 we get

$$
\begin{aligned}
\{p_1, p_2\} &= 0 \\
\{p_1, p_3\} &= -2a_1b_3 \\
\{p_1, p_4\} &= 2a_1b_3 \\
\{p_2, p_3\} &= -4a_2b_3 \\
\{p_2, p_4\} &= -6a_2b_3 \\
\{p_3, p_4\} &= 0 \ .
\end{aligned}
$$

Therefore the matrix P is given by

$$
P = \begin{pmatrix} 0 & 0 & -2a_1b_3 & 2a_1b_3 \\ 0 & 0 & -4a_2b_3 & -6a_2b_3 \\ 2a_1b_3 & 4a_2b_3 & 0 & 0 \\ -2a_1b_3 & 6a_2b_3 & 0 & 0 \end{pmatrix} ,
$$

and P^{-1} is the matrix

$$
P^{-1} = \begin{pmatrix} 0 & 0 & \frac{3}{10a_1b_3} & -\frac{1}{5a_1b_3} \\ 0 & 0 & \frac{1}{10a_2b_3} & \frac{1}{10a_2b_3} \\ -\frac{3}{10a_1b_3} & \frac{1}{5a_1b_3} & 0 & 0 \\ -\frac{1}{10a_2b_3} & -\frac{1}{10a_2b_3} & 0 & 0 \end{pmatrix} .
$$

Using Dirac's formula (29) we obtain a homogeneous quadratic bracket on B_2 given by

$$
\begin{aligned}
\{a_1, a_2\} &= \frac{a_1a_2(3b_3 - b_2 - 2b_1)}{10b_3} \\
\{a_1, b_1\} &= \frac{-a_1(10b_1b_3 - 2b_1b_2 - 3b_1^2 - a_1^2)}{10b_3} \\
\{a_1, b_2\} &= \frac{a_1(10b_2b_3 - 3b_2^2 - 2b_1b_2 - 4a_2^2 - a_1^2)}{10b_3} \\
\{a_1, b_3\} &= \frac{a_1(b_2 - b_1)}{5} \\
\{a_2, b_1\} &= \frac{a_2(2b_1b_3 - 2b_1b_2 + a_1^2)}{10b_3}
\end{aligned}
$$

$$
\begin{aligned}
\{a_2, b_2\} &= \frac{-a_2(8b_2b_3 - 3b_2^2 - 6a_2^2 - 4a_1^2)}{10b_3} \\
\{a_2, b_3\} &= \frac{a_2(b_3 - b_2)}{5} \\
\{b_1, b_2\} &= \frac{10a_1^2b_3 - 3a_1^2b_2 - 2a_2^2b_1 - 3a_1^2b_1}{5b_3} \\
\{b_1, b_3\} &= \frac{2a_1^2}{5} \\
\{b_2, b_3\} &= \frac{2}{5}(a_2^2 - a_1^2) \ .
\end{aligned}
$$

The bracket satisfies the following properties which are analogous to the quadratic A_n Toda (52).

i) It is a homogeneous quadratic Poisson bracket.

ii) It is compatible with the B_2 Lie-Poisson bracket.

iii) The functions $H_n = \frac{1}{n} \operatorname{tr} L^n$ are in involution in this bracket.

iv) We have Lenard type relations $\pi_2 \nabla H_i = \pi_1 \nabla H_{i+1}$ where π_1, π_2 are the Poisson matrices of the linear and quadratic B_2 Toda brackets respectively.

v) The function $\det L$ is the Casimir.

7.2 A recursion operator for Bogoyavlensky–Toda systems of type B_n

In this section, we show that higher polynomial brackets exist also in the case of B_n Toda systems. We will prove that these systems possess a recursion operator and we will construct an infinite sequence of compatible Poisson brackets in which the constants of motion are in involution.

The Hamiltonian for B_n is

$$
H = \frac{1}{2} \sum_{1}^{n} p_j^2 + e^{q_1 - q_2} + \cdots + e^{q_{n-1} - q_n} + e^{q_n} \ . \tag{112}
$$

We make a Flaschka-type transformation

$$
a_i = \frac{1}{2} \, e^{\frac{1}{2}(q_i - q_{i+1})} \ , \qquad\qquad a_n = \frac{1}{2} e^{\frac{1}{2} q_n} \tag{113}
$$

$$
b_i = -\frac{1}{2} p_i \ .
$$

Then

$$
\begin{aligned}
\dot{a}_i &= a_i \, (b_{i+1} - b_i) \quad i = 1, \ldots, n \\
\dot{b}_i &= 2 \, (a_i^2 - a_{i-1}^2) \quad i = 1, \ldots, n \ ,
\end{aligned} \tag{114}
$$

with the convention that $a_0 = b_{n+1} = 0$.

These equations can be written as a Lax pair $\dot{L} = [B, L]$, where L is the symmetric matrix

$$\begin{pmatrix} b_1 & a_1 & & & & & & & \\ a_1 & \ddots & \ddots & & & & & & \\ & \ddots & \ddots & a_{n-1} & & & & & \\ & & a_{n-1} & b_n & a_n & & & & \\ & & & a_n & 0 & -a_n & & & \\ & & & & -a_n & -b_n & \ddots & & \\ & & & & & \ddots & \ddots & -a_1 \\ & & & & & & -a_1 & -b_1 \end{pmatrix} , \tag{115}$$

and B is the skew-symmetric part of L (In the decomposition, lower Borel plus skew-symmetric). We note that the determinant of L is zero.
The mapping $F : \mathbf{R}^{2n} \to \mathbf{R}^{2n}$, $(q_i, p_i) \to (a_i, b_i)$, defined by (113), transforms the standard symplectic bracket into another symplectic bracket π_1 given (up to a constant multiple) by

$$\begin{aligned} \{a_i, b_i\} &= -a_i \\ \{a_i, b_{i+1}\} &= a_i \ . \end{aligned} \tag{116}$$

It is easy to show by induction that

$$\det \pi_1 = a_1^2 a_2^2 \dots a_n^2 \ .$$

The invariant polynomials for B_n, which we denote by

$$H_2, \ H_4, \ \dots \ H_{2n}$$

are defined by $H_{2i} = \frac{1}{2i}$ Tr L^{2i}. The degrees of the first n (independent) polynomials are $2, 4, \dots, 2n$ and the exponents of the corresponding Lie group are $1, 2, \dots, 2n-1$.
We look for a bracket π_3 which satisfies

$$\pi_3 \ \nabla \ H_2 = \pi_1 \ \nabla \ H_4 \ . \tag{117}$$

Using trial and error, we end up with the following homogeneous cubic bracket π_3.

$$\begin{aligned} \{a_i, a_{i+1}\} &= a_i a_{i+1} b_{i+1} \\ \{a_i, b_i\} &= -a_i b_i^2 - a_i^3 \qquad\qquad i = 1, 2, \dots, n-1 \\ \{a_n, b_n\} &= -a_n b_n^2 - 2a_n^3 \\ \{a_i, b_{i+2}\} &= a_i a_{i+1}^2 \\ \{a_i, b_{i+1}\} &= a_i b_{i+1}^2 + a_i^3 \\ \{a_i, b_{i-1}\} &= -a_{i-1}^2 a_i \\ \{b_i, b_{i+1}\} &= 2a_i^2 (b_i + b_{i+1}) \ . \end{aligned} \tag{118}$$

We summarize the properties of this new bracket in the following:

Theorem 12 *The bracket π_3 satisfies*

1. π_3 is Poisson

2. π_3 is compatible with π_1.

3. H_{2i} are in involution.

Define $\mathcal{R} = \pi_3 \pi_1^{-1}$. *Then* $\mathcal{R}$ *is a recursion operator. We obtain a hierarchy*

$$\pi_1, \pi_3, \pi_5, \ldots$$

consisting of compatible Poisson brackets of odd degree in which the constants of motion are in involution.

4. $\pi_{j+2} \nabla H_{2i} = \pi_j \nabla H_{2i+2} \quad \forall\, i,\ j$.

The proof of this result is in [10].

It is interesting to compute the cubic Poisson bracket in (p,q) coordinates. We will see that in the expression for the master symmetry the exponents of the corresponding Lie-group appear explicitly. We reproduce the formula for the cubic Poisson bracket in (p,q)-coordinates from [11].

$$\begin{array}{lcl}
\{q_i, q_{i-1}\} = \{q_i, q_{i-2}\} = \ldots = \{q_i, q_1\} & = & 2p_i \quad i = 2, \ldots, n \\
\{p_i, q_{i-2}\} = \{p_i, q_{i-3}\} = \ldots = \{p_i, q_1\} & = & 2(e^{q_{i-1}-q_i} - e^{q_i - q_{i+1}}) \quad i = 3, \ldots, n-1 \\
\{p_n, q_{n-2}\} = \{p_n, q_{n-3}\} = \ldots = \{p_n, q_1\} & = & 2(e^{q_{n-1}-q_n} - e^{q_n})
\end{array}$$

$$\begin{array}{lcl}
\{q_i, p_i\} & = & p_i^2 + 2e^{q_i - q_{i+1}} \quad i = 1, \ldots, n-1 \\
\{q_n, p_n\} & = & p_n^2 + 2e^{q_n} \\
\{q_{i+1}, p_i\} & = & e^{q_i - q_{i+1}} \\
\{q_i, p_{i+1}\} & = & 2(e^{q_{i+1}-q_{i+2}} - e^{q_i - q_{i+1}}) \quad i = 1, \ldots, n-2 \\
\{q_{n-1}, p_n\} & = & 2e^{q_n} - e^{q_{n-1}-q_n} \\
\{p_i, p_{i+1}\} & = & -e^{q_i - q_{i+1}}(p_i + p_{i+1})
\end{array}$$

In (p,q)-coordinates, J_1 is the (symplectic) canonical Poisson tensor, h_2 is the Hamiltonian , J_3 is the cubic Poisson tensor for B_n and Z_0 is the conformal symmetry for both J_1, J_3 and h_2.

So, with

$$Z_0 = \sum_{i=1}^{n} p_i \frac{\partial}{\partial p_i} + \sum_{i=1}^{n} 2(n-i+1) \frac{\partial}{\partial q_i},$$

we have

$$\mathcal{L}_{Z_0} J_1 = -J_1, \quad \mathcal{L}_{Z_0} J_3 = J_3, \quad \mathcal{L}_{Z_0} h_2 = 2h_2 \ .$$

We obtain a hierarchy of Poisson tensors, master symmetries and invariants which are obtained using Oevel's Theorem. For example, we have

$$[Z_i, \chi_j] = (2j+1)\chi_{i+j} \tag{119}$$

and the coefficients of the first n independent Hamiltonian vector fields correspond to the exponents of a Lie group of type B_n.

7.3 Bi–Hamiltonian formulation of B_n systems

Following the procedure outlined in the introduction we obtain a bi–Hamiltonian formulation of the system. In other words, we define $\pi_{-1} = \mathcal{N}\pi_1 = \pi_1\pi_3^{-1}\pi_1$ and we use it to obtain the desired formulation.

We illustrate with the B_2 Toda system. In this case $\det \pi_1 = a_1^2 a_2^2$ and $\det \pi_3 = a_1^2 a_2^2\, \Delta^2 = \det \pi_1\, \Delta^2$ where

$$\Delta = a_1^4 + 2a_2^2 a_1^2 + 2a_2^2 b_1^2 + b_1^2 b_2^2 - 2a_1^2 b_1 b_2 \ .$$

The explicit formula for π_{-1} is

$$\pi_{-1} = \frac{1}{\Delta} A \ ,$$

where

$$A = \begin{pmatrix} 0 & -a_1a_2b_2 & -a_1(b_2^2+a_1^2+2a_2^2) & a_1(b_1^2+a_1^2+2a_2^2) \\ a_1a_2b_2 & 0 & a_1^2a_2 & -a_2(b_1^2+2a_1^2) \\ a_1(b_2^2+a_1^2+2a_2^2) & -a_1^2a_2 & 0 & -2a_1^2(b_1+b_2) \\ -a_1(b_1^2+a_1^2+2a_2^2) & a_2(b_1^2+2a_1^2) & 2a_1^2(b_1+b_2) & 0 \end{pmatrix} \ .$$

This bracket is Poisson by construction. We will prove later that it is compatible with π_1. We note that $\Delta = \sqrt{\det \mathcal{R}}$ and it is also equal to the product of the non–zero eigenvalues of L. Using the rational bracket π_{-1} we establish the bi-Hamiltonian nature of the system, i.e.

$$\pi_1 \nabla H_2 = \pi_{-1} \nabla H_4 \ .$$

8 C_n TODA SYSTEMS

We now consider C_n Toda systems. We will prove that these systems also possess a recursion operator and we will construct an infinite sequence of compatible Poisson brackets as in the B_n case. We also show that the systems are bi–Hamiltonian.

8.1 A recursion operator for Bogoyavlensky–Toda systems of type C_n

The Hamiltonian for C_n is

$$H = \frac{1}{2}\sum_1^n p_j^2 + e^{q_1-q_2} + \cdots + e^{q_{n-1}-q_n} + e^{2q_n} \ . \tag{120}$$

We make a Flaschka-type transformation

$$a_i = \frac{1}{2}\, e^{\frac{1}{2}(q_i - q_{i+1})} \ , \qquad a_n = \frac{1}{\sqrt{2}} e^{q_n} \tag{121}$$

$$b_i = -\frac{1}{2} p_i \ .$$

The equations in (a, b) coordinates are

$$\begin{array}{rcll} \dot{a}_i & = & a_i\,(b_{i+1} - b_i) & i = 1, \ldots, n-1 \\ \dot{a}_n & = & -2a_n b_n & \\ \dot{b}_i & = & 2\,(a_i^2 - a_{i-1}^2) & i = 1, \ldots, n \ , \end{array} \tag{122}$$

with the convention that $a_0 = 0$.
These equations can be written as a Lax pair $\dot{L} = [B, L]$, where L is the matrix

$$L = \begin{pmatrix} b_1 & a_1 & & & & & & \\ a_1 & \ddots & \ddots & & & & & \\ & \ddots & \ddots & a_{n-1} & & & & \\ & & a_{n-1} & b_n & a_n & & & \\ & & & a_n & -b_n & -a_{n-1} & & \\ & & & & -a_{n-1} & \ddots & \ddots & \\ & & & & & \ddots & \ddots & -a_1 \\ & & & & & & -a_1 & -b_1 \end{pmatrix} , \tag{123}$$

and B is the skew-symmetric part of L.
In the new variables a_i, b_i, the canonical bracket on $\mathbf{R}^{2n}$ is transformed into a bracket π_1 which is given by

$$\begin{array}{lcll} \{a_i, b_i\} & = & -a_i & i = 1, 2, \ldots, n-1 \\ \{a_i, b_{i+1}\} & = & a_i & i = 1, 2, \ldots, n-1 \\ \{a_n, b_n\} & = & -2a_n \ . & \end{array} \tag{124}$$

The invariant polynomials for C_n, which we denote by

$$H_2, \ H_4, \ \ldots \ H_{2n},$$

are defined by $H_{2i} = \frac{1}{2i} \text{ Tr } L^{2i}$.
We look for a bracket π_3 which satisfies

$$\pi_3 \ \nabla \ H_2 = \pi_1 \ \nabla \ H_4 \ . \tag{125}$$

The bracket π_3 was obtained in [12]:

$$\begin{array}{lcll} \{a_i, a_{i+1}\} & = & a_i a_{i+1} b_{i+1} & i = 1, 2, .., n-2 \\ \{a_{n-1}, a_n\} & = & 2a_{n-1} a_n b_n & \\ \{a_i, b_i\} & = & -a_i b_i^2 - a_i^3 & i = 1, 2, \ldots, n-1 \\ \{a_n, b_n\} & = & -2a_n b_n^2 - 2a_n^3 & \\ \{a_i, b_{i+2}\} & = & a_i a_{i+1}^2 & \\ \{a_i, b_{i+1}\} & = & a_i b_{i+1}^2 + a_i^3 & \\ \{a_{n-1}, b_n\} & = & a_{n-1}^3 + a_{n-1} b_n^2 - a_{n-1} a_n^2 & \\ \{a_i, b_{i-1}\} & = & -a_{i-1}^2 a_i & \\ \{a_n, b_{n-1}\} & = & -2a_{n-1}^2 a_n & \\ \{b_i, b_{i+1}\} & = & 2a_i^2 (b_i + b_{i+1}) \ . & \end{array} \tag{126}$$

We summarize the properties of this bracket in the following:

Theorem 13 *The bracket π_3 satisfies*

1. π_3 is Poisson

2. π_3 is compatible with π_1.

3. H_{2i} are in involution.

Define $\mathcal{R} = \pi_3 \pi_1^{-1}$. Then $\mathcal{R}$ is a recursion operator. We obtain a hierarchy

$$\pi_1, \pi_3, \pi_5, \ldots$$

consisting of compatible Poisson brackets of odd degree in which the constants of motion are in involution.

4. $\pi_{j+2} \nabla H_{2i} = \pi_j \nabla H_{2i+2} \qquad \forall\ i,\ j$.

The proofs are precisely the same as in the case of B_n.
Even though it is not necessary to work in (p, q)-coordinates, we reproduce the formulas from [11] for completeness.

$$\begin{array}{lcl}
\{q_i, q_{i-1}\} = \{q_i, q_{i-2}\} = \ldots = \{q_i, q_1\} & = & 2p_i \quad i = 2, \ldots, n \\
\{p_i, q_{i-2}\} = \{p_i, q_{i-3}\} = \ldots = \{p_i, q_1\} & = & 2(e^{q_{i-1}-q_i} - e^{q_i - q_{i+1}}) \quad i = 3, \ldots, n-1 \\
\{p_n, q_{n-2}\} = \{p_n, q_{n-3}\} = \ldots = \{p_n, q_1\} & = & 2e^{q_{n-1}-q_n} - 4e^{2q_n}
\end{array} \tag{127}$$

$$\begin{array}{lcl}
\{q_i, p_i\} & = & p_i^2 + 2e^{q_i - q_{i+1}} \quad i = 1, \ldots, n-1 \\
\{q_n, p_n\} & = & p_n^2 + 2e^{2q_n} \\
\{q_{i+1}, p_i\} & = & e^{q_i - q_{i+1}} \\
\{q_i, p_{i+1}\} & = & 2e^{q_{i+1}-q_{i+2}} - e^{q_i - q_{i+1}} \quad i = 1, \ldots, n-2 \\
\{q_{n-1}, p_n\} & = & 4e^{2q_n} - e^{q_{n-1}-q_n} \\
\{p_i, p_{i+1}\} & = & -e^{q_i - q_{i+1}}(p_i + p_{i+1}) \ .
\end{array} \tag{128}$$

For C_n, the conformal symmetry is

$$Z_0 = \sum_{i=1}^{n} p_i \frac{\partial}{\partial p_i} + \sum_{i=1}^{n} (2n - 2i + 1) \frac{\partial}{\partial q_i},$$

and we have the same constants as in the case of B_n:

$$\mathcal{L}_{Z_0} J_0 = -J_0, \quad \mathcal{L}_{Z_0} J_1 = J_1, \quad \mathcal{L}_{Z_0} H_0 = 2H_0.$$

The relations of Oevel's Theorem are the same of the B_n Toda

$$[Z_i, \chi_j] = (2j+1)\chi_{i+j} \tag{129}$$

$$[Z_i, Z_j] = 2(j-i) Z_{i+j} \tag{130}$$

$$\mathcal{L}_{Z_i} J_j = (2(j-i) - 1) J_{i+j}. \tag{131}$$

Note that (129) gives a method of generating the exponents.

8.2 Bi–Hamiltonian formulation of C_n systems

In order to show that the C_n Toda systems are bi–Hamiltonian we define $\pi_{-1} = \pi_1 \pi_3^{-1} \pi_1$. This is the second bracket required to obtain a bi–Hamiltonian pair. We illustrate with a small dimensional example, namely C_3. The explicit formula for π_{-1} is the following: Let A be the skew–symmetric 6×6 matrix defined by the following terms:

$$
\begin{aligned}
a_{12} &= a_1 a_2 (b_2 a_3^2 + b_3^2 b_2 - a_2^2 b_3 - b_3 b_2^2 + a_2^2 b_2 + b_1^2 b_3) \\
a_{13} &= -2 a_1 a_3 (a_2^2 b_2 + b_1^2 b_3 - b_3 b_2^2) \\
a_{14} &= a_1 (b_2^2 a_3^2 + a_2^4 + b_2^2 b_3^2 - 2 a_2^2 b_2 b_3 + a_1^2 a_3^2 + a_1^2 b_3^2) \\
a_{15} &= -a_1 (2 a_2^4 - 2 a_2^2 b_2 b_3 + a_1^2 a_3^2 + a_1^2 b_3^2 + a_3^2 b_1^2 + b_3^2 b_1^2) \\
a_{16} &= a_1 (2 a_2^4 + 2 a_3^2 b_1^2 + a_2^2 a_1^2 - 2 a_2^2 b_2 b_3 - a_2^2 b_1^2 - 2 b_2^2 a_3^2) \\
a_{23} &= 2 a_2 (b_1^2 + a_1^2) b_3 a_3 \\
a_{24} &= -a_1^2 a_2 (a_3^2 + b_3^2 - 2 b_2 b_3 + a_2^2 - 2 b_1 b_3) \\
a_{25} &= a_2 (2 a_1^2 a_3^2 + 2 a_1^2 b_3^2 - 2 a_1^2 b_2 b_3 + 2 a_2^2 a_1^2 - 2 a_1^2 b_1 b_3 + a_3^2 b_1^2 + b_3^2 b_1^2 + a_2^2 b_1^2) \\
a_{26} &= -a_2 (2 a_1^2 a_3^2 + 2 a_3^2 b_1^2 + a_2^2 b_1^2 + 2 a_2^2 a_1^2 + a_1^4 - 2 a_1^2 b_1 b_2 + b_2^2 b_1^2) \\
a_{34} &= 2 a_1^2 a_3 (a_2^2 - 2 b_1 b_3 - 2 b_2 b_3) \\
a_{35} &= -2 a_3 (2 a_2^2 a_1^2 - 2 a_1^2 b_1 b_3 + a_2^2 b_1^2 - 2 a_1^2 b_2 b_3) \\
a_{36} &= 2 a_3 (2 a_2^2 b_1^2 + 2 a_2^2 a_1^2 + a_1^4 - 2 a_1^2 b_1 b_2 + b_2^2 b_1^2) \\
a_{45} &= 2 a_1^2 (a_3^2 b_1 + b_2 a_3^2 + b_1 b_3^2 + b_3^2 b_2) \\
a_{46} &= 2 a_1^2 (a_2^2 b_1 - 2 a_3^2 b_1 - 2 b_2 a_3^2 - a_2^2 b_3) \\
a_{56} &= 2 (2 a_1^2 a_3^2 b_1 + 2 a_1^2 b_2 a_3^2 + 2 a_1^2 a_2^2 b_3 - 2 a_1^2 a_2^2 b_1 + a_2^2 b_1^2 b_3 + a_2^2 b_1^2 b_2) \ .
\end{aligned}
$$

The Poisson tensor π_{-1} is of the form

$$
\pi_{-1} = \frac{1}{\det L} A \ ,
$$

where

$$
\det L = \sqrt{\det \mathcal{R}} = 2 a_2^2\, b_1^2\, b_2\, b_3 - 2 a_1^2 a_2^2\, b_1\, b_3 - a_3^2\, b_1^2\, b_2^2 + 2 a_3^2\, b_1\, b_2\, a_1^2 - a_1^4 a_3^2 - b_1^2\, b_2^2\, b_3^2 + 2 a_1^2 b_1\, b_2\, b_3^2 - a_1^4\, b_3^2 - a_2^4 b_1^2 \ .
$$

As in the case of B_2 we have

$$
\pi_1 \nabla H_2 = \pi_{-1} \nabla H_4 \ .
$$

9 D_n TODA SYSTEMS

In this section, we show that higher polynomial brackets exist also in the case of D_n Bogoyavlensky-Toda systems. Using Flaschka coordinates, we will prove that these systems possess a recursion operator and we will construct an infinite sequence of compatible Poisson brackets in which the constants of motion are in involution. We also show that the system is bi–Hamiltonian.

9.1 A recursion operator for D_n Bogoyavlensky-Toda systems in Flaschka coordinates

The Hamiltonian for D_n is

$$
H = \frac{1}{2} \sum_{1}^{n} p_j^2 + e^{q_1 - q_2} + \cdots + e^{q_{n-1} - q_n} + e^{q_{n-1} + q_n} \qquad n \geq 4 \ . \tag{132}
$$

We make a Flaschka-type transformation, $F : \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ defined by

$$F : (q_1, \ldots, q_n, p_1, \ldots, p_n) \to (a_1, \ldots, a_n, b_1, \ldots, b_n) \ ,$$

with

$$a_i = \frac{1}{2} e^{\frac{1}{2}(q_i - q_{i+1})} \ , \qquad i = 1, 2, \ldots, n-1, \qquad a_n = \frac{1}{2} e^{\frac{1}{2}(q_{n-1} + q_n)} \ , \tag{133}$$

$$b_i = -\frac{1}{2} p_i, \qquad i = 1, 2, \ldots, n \ .$$

Then

$$\begin{array}{lcll} \dot{a}_i & = & a_i \left(b_{i+1} - b_i\right) & i = 1, 2, \ldots, n-1 \\ \dot{a}_n & = & -a_n (b_{n-1} + b_n) & \\ \dot{b}_i & = & 2 \left(a_i^2 - a_{i-1}^2\right) & i = 1, 2, \ldots, n-2 \text{ and } i = n \\ \dot{b}_{n-1} & = & 2(a_n^2 + a_{n-1}^2 - a_{n-2}^2) \ . & \end{array} \tag{134}$$

These equations can be written as a Lax pair $\dot{L} = [B, L]$, where L is the symmetric matrix

$$\begin{pmatrix} b_1 & a_1 & & & & & & & \\ a_1 & \ddots & \ddots & & & & & & \\ & \ddots & \ddots & a_{n-1} & -a_n & 0 & & & \\ & & a_{n-1} & b_n & 0 & a_n & & & \\ & & -a_n & 0 & -b_n & -a_{n-1} & & & \\ & & 0 & a_n & -a_{n-1} & \ddots & \ddots & & \\ & & & & & \ddots & \ddots & -a_1 & \\ & & & & & & -a_1 & -b_1 \end{pmatrix} , \tag{135}$$

and B is the skew-symmetric part of L (In the decomposition, lower Borel plus skew-symmetric).

The mapping $F : \mathbf{R}^{2n} \to \mathbf{R}^{2n}$, $(q_i, p_i) \to (a_i, b_i)$, defined by (133), transforms the standard symplectic bracket J_0 into another symplectic bracket π_1 given (up to a constant multiple) by

$$\begin{array}{lcll} \{a_i, b_i\} & = & -\frac{1}{2} a_i & i = 1, 2, \ldots, n \\ \{a_i, b_{i+1}\} & = & \frac{1}{2} a_i & i = 1, 2, \ldots, n-1 \\ \{a_n, b_{n-1}\} & = & -\frac{1}{2} a_n . & \end{array} \tag{136}$$

We obtain a hierarchy of invariant polynomials, which we denote by

$$H_2, \ H_4, \ \ldots, \ H_{2n}, \ldots$$

defined by $H_{2i} = \frac{1}{2i}$ Tr L^{2i}. The degrees of the first $n-1$ (independent) polynomials are $2, 4, \ldots, 2n-2$. We also define

$$P_n = \sqrt{\det L} \ .$$

The degree of P_n is n. The set $\{H_2, \ H_4, \ \ldots, \ H_{2n-2}, P_n\}$ corresponds to the Chevalley invariants for a Lie group of type D_n. The exponents of the Lie group is the set

$\{1, 3, 5, \ldots, 2n-3, n-1\}$ which is obtained by subtracting 1 from the degrees of the invariant polynomials. A conjecture of Flaschka states that the degrees of the Poisson brackets is in one–to–one correspondence with the exponents of the corresponding Lie group.
Taking $H_2 = \frac{1}{2}\mathrm{Tr}\ L^2$ as the Hamiltonian we have that

$$\pi_1 \nabla H_2$$

gives precisely equations (134). In this section, we find a bracket π_{-1} which satisfies

$$\pi_1 \nabla H_2 = \pi_{-1} \nabla H_4 \ .$$

First, we define a bracket π_3 which satisfies

$$\pi_3 \ \nabla \ H_2 = \pi_1 \ \nabla \ H_4 \ , \tag{137}$$

and whose non-zero terms are

$$\begin{array}{lcll}
\{a_i, a_{i+1}\} & = & a_i a_{i+1} b_{i+1} & i = 1, 2, \ldots, n-2 \\
\{a_{n-2}, a_n\} & = & a_{n-2} a_n b_{n-1} & \\
\{a_{n-1}, a_n\} & = & 2a_{n-1} a_n b_n & \\
\{a_i, b_i\} & = & -a_i(b_i^2 + a_i^2) & i = 1, 2, \ldots, n-2 \\
\{a_{n-1}, b_{n-1}\} & = & -a_{n-1}(a_{n-1}^2 + 3a_n^2 + b_{n-1}^2) & \\
\{a_n, b_n\} & = & -a_n(a_n^2 + b_n^2 - a_{n-1}^2) & \\
\{a_i, b_{i+1}\} & = & a_i(a_i^2 + b_{i+1}^2) & i = 1, 2, \ldots n-2 \\
\{a_{n-1}, b_n\} & = & a_{n-1}(a_{n-1}^2 + b_n^2 - a_n^2) & \\
\{a_i, b_{i+2}\} & = & a_i a_{i+1}^2 & i = 1, 2, \ldots, n-3 \\
\{a_{n-2}, b_n\} & = & a_{n-2}(a_{n-1}^2 - a_n^2) & \\
\{a_i, b_{i-1}\} & = & -a_{i-1}^2 a_i & i = 2, 3, \ldots, n-1 \\
\{a_n, b_{n-2}\} & = & -a_{n-2}^2 a_n & \\
\{a_n, b_{n-1}\} & = & -a_n(3a_{n-1}^2 + a_n^2 + b_{n-1}^2) & \\
\{b_i, b_{i+1}\} & = & 2a_i^2(b_i + b_{i+1}) & i = 1, 2, \ldots, n-2 \\
\{b_{n-1}, b_n\} & = & 2a_{n-1}^2(b_{n-1} + b_n) + 2a_n^2(b_n - b_{n-1}) \ . &
\end{array} \tag{138}$$

This bracket appeared recently in [13]. We summarize the properties of this new bracket in the following:

Theorem 14 *The bracket π_3 satisfies*
1. π_3 is Poisson.
2. π_3 is compatible with π_1.
Define $\mathcal{R} = \pi_3 \pi_1^{-1}$. Then $\mathcal{R}$ is a recursion operator. We obtain a hierarchy

$$\pi_1, \pi_3, \pi_5, \ldots$$

consisting of compatible Poisson brackets of odd degree in which the constants of motion are in involution.
3. All the H_{2i} and P_n are in involution with respect to all the brackets $\pi_1, \pi_3, \pi_5, \ldots$.
4. $\pi_{j+2} \ \nabla \ H_{2i} = \pi_j \ \nabla \ H_{2i+2} \qquad \forall \ i, \ j$.

The proof of *1.* is a straightforward verification of the Jacobi identity. We will see later, in the next subsection, that π_3 is the Lie derivative of π_1 in the direction of a master symmetry and this fact makes π_1, π_3 compatible. *4.* follows from properties of the recursion operator. *3.* is a consequence of *4* (see for example Proposition 3 for a method of proof). The only part which is not obvious is the involution of P_n with H_n which will be proved at the end of next subsection using master symmetries.

9.2 Master symmetries

We would like to make some observations concerning master symmetries.
Due to the presence of a recursion operator, we will use the approach of Oevel. We define Z_0 to be the Euler vector field

$$Z_0 = \sum_{i=1}^{n} a_i \frac{\partial}{\partial a_i} + b_i \frac{\partial}{\partial b_i} \ .$$

We define the master symmetries Z_i by:

$$Z_i = \mathcal{R}^i Z_0 \ .$$

For obvious reasons we use the notations

$$X_{2i} = Z_i \ , h_i = H_{2i+2} \ , \Pi_i = \pi_{2i+1} \ , \psi_i = \chi_{2i+2} \ , i = 0, 1, 2, \dots$$

where χ_{2i} denotes the Hamiltonian vector field generated by H_{2i}, with respect to π_1. This notation is convenient since X_2 is a master symmetry which raises the degrees of invariants and Poisson tensors by 2 each time. One calculates easily that

$$\mathcal{L}_{Z_0}\Pi_0 = -\Pi_0 \ , \quad \mathcal{L}_{Z_0}\Pi_1 = \Pi_1 \ , \quad \mathcal{L}_{Z_0}h_0 = 2h_0 \ .$$

Therefore Z_0 is a conformal symmetry for Π_0, Π_1 and h_0. The constants appearing in Oevel's Theorem are $\lambda = -1$, $\mu = 1$ and $\nu = 2$. Therefore we obtain

$$[Z_i, \psi_j] = (1 + 2j)\, \psi_{i+j} \Longleftrightarrow [X_{2i}, \chi_{2j+2}] = (1 + 2j)\, \chi_{2(i+j+1)} \tag{139}$$

$$[Z_i, Z_j] = 2\,(j - i)\, Z_{i+j} \Longleftrightarrow [X_{2i}, X_{2j}] = 2\,(j - i)\, X_{2(i+j)} \tag{140}$$

$$\mathcal{L}_{Z_i}\,(\Pi_j) = (2j - 2i - 1)\, \Pi_{i+j} \Longleftrightarrow \mathcal{L}_{X_{2i}}\,(\pi_{2j+1}) = (2j - 2i - 1)\, \pi_{2(i+j)+1} \tag{141}$$

$$Z_i\,(h_j) = (2 + 2i + 2j)\, h_{i+j} \Longleftrightarrow X_{2i}\,(H_{2j}) = 2\,(i + j)\, H_{2(i+j)} \ . \tag{142}$$

Remark 1: The relation (141) implies that $\mathcal{L}_{X_2}(\pi_1) = -3\pi_3$ and therefore π_3 is Lie-derivative of π_1 in the direction of a master symmetry. This makes π_1 compatible with π_3 (see Lemma 1).
Remark 2: The relation (139) gives a procedure for generating almost all the exponents. As we mentioned in the introduction, the last exponent is generated by the application of the conformal symmetry on the Hamiltonian vector field corresponding to the Pfaffian of the Jacobi matrix.

It is interesting to note that one can obtain the master symmetry X_2 by using the matrix equation

$$\dot{L} = [B, L] + L^3 \ , \tag{143}$$

where L is the Lax matrix (135) and B is the skew-symmetric matrix defined as follows:

$$
B = \left(\begin{array}{cccccccccccc}
0 & x_1 & y_1 & 0 & & & & & & & & \\
-x_1 & 0 & x_2 & y_2 & \ddots & & & & & & & \\
-y_1 & -x_2 & 0 & \ddots & \ddots & 0 & & & & 0 & & \\
0 & -y_2 & \ddots & \ddots & x_{n-2} & y_{n-2} & y_{n-1} & & & & & \\
 & \ddots & \ddots & -x_{n-2} & 0 & x_{n-1} & x_n & 0 & & & & \\
 & & 0 & -y_{n-2} & -x_{n-1} & 0 & 0 & -x_n & -y_{n-1} & & & \\
 & & & -y_{n-1} & -x_n & 0 & 0 & -x_{n-1} & -y_{n-2} & 0 & & \\
 & & & & 0 & x_n & x_{n-1} & 0 & -x_{n-2} & \ddots & \ddots & \\
 & & & & & y_{n-1} & y_{n-2} & x_{n-2} & \ddots & \ddots & -y_2 & 0 \\
 & & 0 & & & & 0 & \ddots & \ddots & 0 & -x_2 & -y_1 \\
 & & & & & & & \ddots & y_2 & x_2 & 0 & -x_1 \\
 & & & & & & & & 0 & y_1 & x_1 & 0
\end{array} \right)
$$

where

$$
\begin{aligned}
x_i &= a_i \left\{ \sum_{j=1}^{i-1} b_j + (i+1-n)\,(b_i + b_{i+1}) \right\}, \quad i = 1, 2, \ldots, n-1 \\
x_n &= -a_n \sum_{j=1}^{n-2} b_j \\
y_i &= (i+1-n)\, a_i a_{i+1}, \qquad\quad i = 1, 2, \ldots, n-2 \\
y_{n-1} &= a_{n-2} a_n .
\end{aligned}
$$

It is interesting to note that the y_i is a constant times a product of $a_j a_k$ where the product is determined from the Dynkin diagram of a Lie algebra of type D_n.
The matrix B was chosen in such a way that both sides of (143) have the same form. The components of the vector field X_2 are defined by the right hand side of (143).
Finally we note the action of the first master symmetry on $P_n = \sqrt{\det L}$:

$$
X_2(P_n) = P_n H_2 \ .
$$

Remark: This last result should be expected since the eigenvalues of L satisfy $\dot{\lambda} = \lambda^3$ under (143). Therefore,

$$
\begin{aligned}
X_2(P_n) &= X_2(\sqrt{\det L}) \\
&= X_2\left(\sqrt{\lambda_1 \ldots \lambda_n}\right) \\
&= \frac{1}{2}\,(\lambda_1 \ldots \lambda_n)^{-\frac{1}{2}} \left(\dot{\lambda}_1 \lambda_2 \ldots \lambda_n + \lambda_1 \dot{\lambda}_2 \ldots \lambda_n + \ldots + \lambda_1 \ldots \dot{\lambda}_n\right) \\
&= \frac{1}{2\sqrt{\det L}} \left(\lambda_1^3 \lambda_2 \ldots \lambda_n + \lambda_1 \lambda_2^3 \ldots \lambda_n + \ldots + \lambda_1 \ldots \lambda_n^3\right) \\
&= \frac{\det L}{2\sqrt{\det L}} \left(\lambda_1^2 + \lambda_2^2 + \ldots + \lambda_n^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \sqrt{\det L}\;\frac{(\lambda_1^2+\lambda_2^2+\ldots+\lambda_n^2)}{2} \\
&= P_n H_2\ .
\end{aligned}
$$

We conclude this subsection by proving the involution of H_i with P_n. It is clearly enough to show the involution of the eigenvalues of L since P_n and H_i are both functions of the eigenvalues. It is well–known that the eigenvalues are in involution with respect to the symplectic bracket π_1. We will give here a proof based on the Lenard relations (137). Let λ and μ be two distinct eigenvalues and let U, V be the gradients of λ and μ respectively. We use the notation $\{\ ,\ \}$ to denote the bracket π_1 and $\langle\ ,\ \rangle$ the standard inner product. The Lenard relations (137) translate into $\pi_3\, U = \lambda^2\, \pi_1\, U$ and $\pi_3\, V = \mu^2\, \pi_1\, V$. Therefore,

$$
\begin{aligned}
\{\lambda,\mu\} &= \langle \pi_1 U, V\rangle \\
&= \frac{1}{\lambda^2}\,\langle \pi_3 U, V\rangle \\
&= -\frac{1}{\lambda^2}\,\langle U, \pi_3 V\rangle \\
&= -\frac{1}{\lambda^2}\,\left\langle U, \mu^2\pi_1 V\right\rangle \\
&= -\frac{\mu^2}{\lambda^2}\,\langle U, \pi_1 V\rangle \\
&= \frac{\mu^2}{\lambda^2}\,\langle \pi_1 U, V\rangle \\
&= \frac{\mu^2}{\lambda^2}\{\lambda,\mu\}\ .
\end{aligned}
$$

Therefore, $\{\lambda,\mu\}=0$. To show the involution with respect to all brackets π_{2j+1} and in view of (141) it is enough to show the following: Let f_1, f_2 be two functions in involution with respect to the Poisson bracket π, let X be a vector field such that $X(f_i)=f_i^3$ for $i=1,2$. Define a Poisson bracket w by $w=\mathcal{L}_X\pi$. Then the functions f_1, f_2 remain in involution with respect to the bracket w. The proof follows trivially if we write $w=\mathcal{L}_X\pi$ in Poisson form

$$
\{f_1,f_2\}_w = X\{f_1,f_2\}_\pi - \{f_1, X(f_2)\}_\pi - \{X(f_1), f_2\}_\pi\ .
$$

Remark: We have to point out that unlike the case of B_n and C_n the cubic bracket (138) was discovered not by manipulating the left hand side of (137) but through the use of the master symmetry X_2. In other words, we construct the master symmetry X_2 using (143) and then compute $\pi_3 = -\frac{1}{3}\mathcal{L}_{X_2}\pi_1$.

9.3 A recursion operator for D_n Toda systems in natural (q,p) coordinates

We now define a bi-Hamiltonian formulation for D_n Bogoyavlensky-Toda systems in natural (q_i,p_i) coordinates. This bracket is simply the pull-back of π_3 under the Flaschka transformation (133). After some tedious calculation, we obtain the following bracket in (q_i,p_i) coordinates:

$$
\begin{array}{lcll}
\{q_i,q_j\} & = & -2p_j\ , \quad i<j & \\
\{q_i,p_i\} & = & p_i^2+2e^{q_i-q_{i+1}} & i=1,2,\ldots,n-2 \\
\{q_{n-1},p_{n-1}\} & = & p_{n-1}^2+2e^{q_{n-1}-q_n}+2e^{q_{n-1}+q_n} & \\
\{q_n,p_n\} & = & p_n^2 & \\
\{q_i,p_{i-1}\} & = & e^{q_{i-1}-q_i} & i=2,3,\ldots,n-1 \\
\{q_n,p_{n-1}\} & = & e^{q_{n-1}-q_n}-e^{q_{n-1}+q_n} & \\
\{q_i,p_{i+1}\} & = & -e^{q_i-q_{i+1}}+2e^{q_{i+1}-q_{i+2}} & i=1,2,\ldots,n-3 \\
\{q_{n-2},p_{n-1}\} & = & -e^{q_{n-2}-q_{n-1}}+2e^{q_{n-1}-q_n}+2e^{q_{n-1}+q_n} & \\
\{q_{n-1},p_n\} & = & -e^{q_{n-1}-q_n}+e^{q_{n-1}+q_n} & \\
\{q_i,p_j\} & = & -2e^{q_{j-1}-q_j}+2e^{q_j-q_{j+1}} & 1\le i<j-1\le n-3 \\
\{q_i,p_{n-1}\} & = & -2e^{q_{n-2}-q_{n-1}}+2e^{q_{n-1}-q_n}+2e^{q_{n-1}+q_n} & i=1,2,\ldots,n-3 \\
\{q_i,p_n\} & = & -2e^{q_{n-1}-q_n}+2e^{q_{n-1}+q_n} & i=1,2,\ldots,n-2 \\
\{p_i,p_{i+1}\} & = & -e^{q_i-q_{i+1}}(p_i+p_{i+1}) & i=1,2,\ldots,n-2 \\
\{p_{n-1},p_n\} & = & -(p_{n-1}+p_n)e^{q_{n-1}-q_n}+(p_{n-1}-p_n)e^{q_{n-1}+q_n}\ ; &
\end{array}
\tag{144}
$$

and all other brackets are zero.
Denote this Poisson tensor by J_1 and let J_0 be the standard symplectic bracket. A simple computation leads to the following:

Theorem 15 *The bracket J_1 satisfies*

1. *J_1 is Poisson.*
2. *J_1 is compatible with J_0.*
3. *The mapping F given by (133) is a Poisson mapping between J_1 and the cubic bracket π_3.*

Thus, in (q,p) coordinates we also have a non-degenerate pair (J_0,J_1) for D_n Bogoyavlensky-Toda and therefore we may define a recursion operator $\mathcal{R}=J_1J_0^{-1}$. We obtain a hierarchy of mutually compatible Poisson tensors defined by $J_i=\mathcal{R}^iJ_0$.
The vector field

$$
Z_0=\sum_{i=1}^n p_i\frac{\partial}{\partial p_i}+\sum_{i=1}^n 2(n-i)\frac{\partial}{\partial q_i}, \tag{145}
$$

is a conformal symmetry for the Poisson tensors J_0 and J_1 and for the Hamiltonian

$$
h_0=\frac{1}{2}\sum_1^n p_j^2+e^{q_1-q_2}+\cdots+e^{q_{n-1}-q_n}+e^{q_{n-1}+q_n}\ . \tag{146}
$$

We compute

$$
\mathcal{L}_{Z_0}J_0=-J_0\ ,\qquad \mathcal{L}_{Z_0}J_1=J_1\ ,\qquad \mathcal{L}_{Z_0}h_0=2h_0\ . \tag{147}
$$

So Oevel's Theorem applies. With $Z_i=\mathcal{R}^iZ_0$, $\chi_0=\chi_{h_0}$ and $\chi_i=\mathcal{R}^i\chi_0$ one calculates easily that

$$
\begin{array}{rrcl}
(a) & [Z_i,\chi_j] & = & (1+2j)\,\chi_{i+j} \\
(b) & [Z_i,Z_j] & = & 2\,(j-i)\,Z_{i+j} \\
(c) & \mathcal{L}_{Z_i}(J_j) & = & (2j-2i-1)\,J_{i+j}\ .
\end{array}
$$

Note that (a) gives the exponents (except one) for a Lie group of type D_n.

The action of the first master symmetry on P_n is the same as in Flaschka coordinates:

$$Z_1\,(P_n) = h_0\,P_n\ . \tag{148}$$

Finally, we calculate that

$$[Z_0, \chi_{P_n}] = (n-1)\,\chi_{P_n}\ , \tag{149}$$

producing the last exponent.

9.4 Bi–Hamiltonian formulation of Bogoyavlensky–Toda systems of type D_n

In order to show that the D_n Toda systems are bi–Hamiltonian we use the same procedure as in the previous two cases. The tensors π_1 and π_3 are both invertible and we define $\pi_{-1} = \pi_1\pi_3^{-1}\pi_1$. This is the second bracket required to obtain a bi–Hamiltonian pair. We illustrate with a small dimensional example, namely D_4. The explicit formula for π_{-1} is the following:

Let A be a skew–symmetric 8×8 matrix given by the following terms:

$$\begin{aligned} a_{12} = -a_1a_2(&a_2^2a_3^2b_4 + b_2b_3^2b_4^2 + 2b_4b_3b_2a_4^2 - 2b_4b_3b_2a_3^2 + b_2a_3^4 + a_4^4b_2 - 2b_2a_4^2a_3^2 + a_2^2b_2b_4^2 \\ &-b_2^2b_3b_4^2 + b_4b_2^2a_3^2 - b_4b_2^2a_4^2 + b_1^2b_3b_4^2 + b_1^2a_4^2b_4 - b_1^2a_3^2b_4 - a_2^2b_3b_4^2 - a_2^2a_4^2b_4) \end{aligned}$$

$$\begin{aligned} a_{13} = -a_1a_3(&b_2^2b_3b_4^2 - b_4b_2^2a_3^2 - a_2^2b_2b_4^2 + b_4b_2^2a_4^2 - b_1^2b_3b_4^2 - b_1^2a_4^2b_4 + b_1^2a_3^2b_4 - a_2^4b_4 + 2a_2^2b_2b_3b_4 \\ &-a_1^2a_2^2b_4 + a_2^2b_2a_4^2 - a_2^2a_3^2b_2 - b_2^2b_3^2b_4 + a_3^2b_2^2b_3 - b_3b_2^2a_4^2 + b_1^2a_2^2b_4 + b_1^2b_3^2b_4 - b_1^2a_3^2b_3 + b_1^2b_3a_4^2) \end{aligned}$$

$$\begin{aligned} a_{14} = a_1a_4(&a_2^2b_2b_4^2 - b_2^2b_3b_4^2 + b_4b_2^2a_3^2 - b_4b_2^2a_4^2 + b_1^2b_3b_4^2 + b_1^2a_4^2b_4 - b_1^2a_3^2b_4 - a_2^4b_4 + 2a_2^2b_2b_3b_4 - a_1^2a_2^2b_4 \\ &+a_2^2b_2a_4^2 - a_2^2a_3^2b_2 - b_2^2b_3^2b_4 + a_3^2b_2^2b_3 - b_3b_2^2a_4^2 + b_1^2a_2^2b_4 + b_1^2b_3^2b_4 - b_1^2a_3^2b_3 + b_1^2b_3a_4^2) \end{aligned}$$

$$\begin{aligned} a_{15} = -a_1(&2a_4^2b_3b_2^2b_4 + 2b_4a_2^2a_3^2b_2 + b_2^2a_4^4 + a_2^4b_4^2 - 2b_3b_2a_2^2b_4^2 + b_2^2a_3^4 - 2b_3b_4b_2^2a_3^2 - 2b_2^2a_4^2a_3^2 - 2b_4a_2^2a_4^2b_2 \\ &+b_3^2b_4^2b_2^2 + a_1^2b_3^2b_4^2 - 2a_1^2b_3b_4a_3^2 + 2a_1^2b_3b_4a_4^2 + a_1^2a_4^4 - 2a_1^2a_4^2a_3^2 + a_1^2a_3^4) \end{aligned}$$

$$\begin{aligned} a_{16} = a_1(&2a_1^2b_3b_4a_4^2 + 2b_1^2b_3b_4a_4^2 - 2a_1^2b_3b_4a_3^2 - 2b_4a_2^2a_4^2b_2 + 2b_4a_2^2a_3^2b_2 + 2a_2^4b_4^2 - 2b_3b_2a_2^2b_4^2 \\ &+a_1^2a_4^4 + a_1^2a_3^4 + b_1^2a_4^4 + b_1^2a_3^4 - 2a_1^2a_4^2a_3^2 - 2b_1^2a_3^2a_4^2 + a_1^2b_3^2b_4^2 - 2b_1^2a_3^2b_3b_4 + b_1^2b_3^2b_4^2) \end{aligned}$$

$$\begin{aligned} a_{17} = a_1(&2b_2^2a_4^4 - 2b_1^2b_3b_4a_4^2 + 2b_2^2a_3^4 + 2a_4^2b_3b_2^2b_4 - 2a_2^4b_4^2 - 4b_2^2a_4^2a_3^2 + 2b_3b_2a_2^2b_4^2 - 2b_3b_4b_2^2a_3^2 \\ &-2b_1^2a_4^4 - 2b_1^2a_3^4 + 4b_1^2a_3^2a_4^2 + 2b_1^2a_3^2b_3b_4 + b_1^2a_2^2b_4^2 - a_1^2a_2^2b_4^2) \end{aligned}$$

$$\begin{aligned} a_{18} = a_1(&a_1^2a_2^2a_4^2 - a_1^2a_2^2a_3^2 - 2b_1^2b_3b_4a_4^2 + 2b_2^2a_4^4 - 2b_4a_2^2a_4^2b_2 - 2b_2^2a_3^4 + 2a_4^2b_3b_2^2b_4 - 2b_4a_2^2a_3^2b_2 \\ &+2b_3b_4b_2^2a_3^2 - 2b_1^2a_4^4 + 2b_1^2a_3^4 - 2b_1^2a_3^2b_3b_4 + b_1^2a_2^2a_3^2 - b_1^2a_2^2a_4^2) \end{aligned}$$

$$a_{23} = a_2a_3(b_1^2a_3^2b_4 - b_1^2b_3b_4^2 - b_1^2a_4^2b_4 - b_3a_1^2b_4^2 - a_1^2a_4^2b_4 + a_1^2a_3^2b_4 + b_1^2b_3^2b_4 - b_1^2a_3^2b_3 + b_1^2b_3a_4^2 \\ +b_3^2b_4a_1^2 + a_4^2b_3a_1^2 - b_3a_1^2a_3^2 - b_4b_1^2b_2^2 + 2b_4b_1a_1^2b_2 - a_1^2a_2^2b_4 - b_4a_1^4)$$

$$a_{24} = -a_2a_4(b_1^2b_3b_4^2 + b_1^2a_4^2b_4 - b_1^2a_3^2b_4 + b_3a_1^2b_4^2 + a_1^2a_4^2b_4 - a_1^2a_3^2b_4 + b_1^2b_3^2b_4 - b_1^2a_3^2b_3 + b_1^2b_3a_4^2 \\ +b_3^2b_4a_1^2 + a_4^2b_3a_1^2 - b_3a_1^2a_3^2 - b_4b_1^2b_2^2 + 2b_4b_1a_1^2b_2 - a_1^2a_2^2b_4 - b_4a_1^4)$$

$$a_{25} = -a_1^2a_2(2b_4b_3a_3^2 - b_3^2b_4^2 - 2b_4a4^2b_3 - a_4^4 + 2a_4^2a_3^2 - a_3^4 + 2b_1b_3b_4^2 - 2b_4b_1a_3^2 + 2b_4a_4^2b_1 - a_2^2b_4^2 \\ +2b_3b_2b_4^2 + 2b_4b_2a_4^2 - 2b_4a_3^2b_2)$$

$$a_{26} = -a_2(4a_1^2b_3b_4a_4^2 + 2b_1^2b_3b_4a_4^2 - 2a_1^2b_4b_2a_4^2 - 2a_1^2b_3b_2b_4^2 + 2a_1^2b_4a_3^2b_2 - 2b_1a_1^2a_4^2b_4 + 2b_1a_1^2a_3^2b_4 \\ -2b_1b_3a_1^2b_4^2 - 4a_1^2b_3b_4a_3^2 + 2a_1^2a_4^4 + 2a_1^2a_3^4 + b_1^2a_4^4 + b_1^2a_3^4 - 4a_1^2a_4^2a_3^2 - 2b_1^2a_3^2a_4^2 + 2a_1^2b_3^2b_4^2 - \\ 2b_1^2a_3^2b_3b_4 + b_1^2a_2^2b_4^2 + 2a_1^2a_2^2b_4^2 + b_1^2b_3^2b_4^2)$$

$$a_{27} = a_2(a_1^4b_4^2 + 2a_1^2b_3b_4a_4^2 + 2b_1^2b_3b_4a_4^2 - 2a_1^2b_3b_4a_3^2 + 2a_1^2a_4^4 + 2a_1^2a_3^4 + 2b_1^2a_4^4 + 2b_1^2a_3^4 - 4a_1^2a_4^2a_3^2 \\ -4b_1^2a_3^2a_4^2 - 2b_1^2a_3^2b_3b_4 + b_1^2a_2^2b_4^2 + 2a_1^2a_2^2b_4^2 - 2b_1a_1^2b_2b_4^2 + b_1^2b_2^2b_4^2)$$

$$a_{28} = -a_2(a_4^2b_1^2b_2^2 - 2a_1^2b_3b_4a_4^2 - 2b_1^2b_3b_4a_4^2 - 2a_1^2b_3b_4a_3^2 - 2a_1^2a_4^4 + 2a_1^2a_3^4 - 2b_1^2a_4^4 + 2b_1^2a_3^4 \\ -2b_1^2a_3^2b_3b_4 - 2b_1a_4^2a_1^2b_2 - b_1^2a_3^2b_2^2 + b_1^2a_2^2a_3^2 - b_1^2a_2^2a_4^2 - a_1^4a_3^2 + a_1^4a_4^2 + 2b_1a_1^2a_3^2b_2)$$

$$a_{34} = -2a_3a_4b_4(b_1^2a_2^2 + b_1^2b_2^2 - 2b_1a_1^2b_2 + a_1^2a_2^2 + a_1^4)$$

$$a_{35} = -a_3a_1^2(2b_4b_1a_3^2 - 2b_1b_3b_4^2 - 2b_4a_4^2b_1 + a_2^2b_4^2 - 2b_3b_2b_4^2 - 2b_4b_2a_4^2 + 2b_4a_3^2b_2 + 2b_1b_4a_2^2 + \\ 2b_1b_3^2b_4 - 2b_1a_3^2b_3 + 2a_4^2b_1b_3 - 2a_2^2b_3b_4 + a_2^2a_3^2 - a_2^2a_4^2 + 2b_2b_3^2b_4 + 2a_4^2b_2b_3 - 2a_3^2b_2b_3)$$

$$a_{36} = -a_3(2b_1^2b_4a_2^2b_2 - 4b_1a_1^2a_2^2b_4 + 2a_1^2a_2^2a_4^2 - 2a_1^2a_2^2a_3^2 + 2a_1^2b_4b_2a_4^2 + 2a_1^2b_3b_2b_4^2 - 2a_1^2a_4^2b_2b_3 \\ +2a_1^2a_3^2b_2b_3 + 2a_2^2b_1^2b_3b_4 + 2a_1^2b_1a_3^2b_3 + 4a_1^2a_2^2b_3b_4 - 2a_1^2b_2b_3^2b_4 - 2a_1^2a_4^2b_1b_3 - 2a_1^2b_1b_3^2b_4 - 2a_1^2b_4a_3^2b_2 \\ +2b_1a_1^2a_4^2b_4 - 2b_1a_1^2a_3^2b_4 + 2b_1b_3a_1^2b_4^2 - b_1^2a_2^2b_4^2 - 2a_1^2a_2^2b_4^2 - b_1^2a_2^2a_3^2 + b_1^2a_2^2a_4^2)$$

$$a_{37} = a_3(2b_1^2b_4a_2^2b_2 - 2b_1a_1^2a_2^2b_4 + 2a_1^2a_2^2a_4^2 - a_1^4b_4^2 - 2a_1^2a_2^2a_3^2 + 2a_2^2b_1^2b_3b_4 + 2a_1^2a_2^2b_3b_4 + a_4^2b_1^2b_2^2 \\ -2b_1^2a_2^2b_4^2 - 2a_1^2a_2^2b_4^2 + 2b_1a_1^2b_2b_4^2 - 2b_1a_4^2a_1^2b_2 - b_1^2a_3^2b_2^2 - 2b_1^2a_2^2a_3^2 + 2b_1^2a_2^2a_4^2 - b_1^2b_2^2b_4^2 - a_1^4a_3^2 \\ +a_1^4a_4^2 + 2b_1a_1^2a_3^2b_2)$$

$$a_{38} = a_3(2b_1a_1^2a_2^2b_3 + 2a_1^2a_2^2a_4^2 + 2a_1^2a_2^2a_3^2 + a_1^4b_3^2 + b_1^2b_2^2b_3^2 + 3a_4^2b_1^2b_2^2 - 6b_1a_4^2a_1^2b_2 + b_1^2a_3^2b_2^2 + b_1^2a_2^4 \\ +2b_1^2a_2^2a_3^2 + 2b_1^2a_2^2a_4^2 - 2b_1^2a_2^2b_3b_2 - 2b_1a_1^2b_2b_3^2 + a_1^4a_3^2 + 3a_1^4a_4^2 - 2b_1a_1^2a_3^2b_2)$$

$$a_{45} = a_4a_1^2(2b_1b_3b_4^2 - 2b_4b_1a_3^2 + 2b_4a_4^2b_1 - a_2^2b_4^2 + 2b_3b_2b_4^2 + 2b_4b_2a_4^2 - 2b_4a_3^2b_2 + 2b_1b_4a_2^2 + 2b_1b_3^2b_4 - \\ 2b_1a_3^2b_3 + 2a_4^2b_1b_3 - 2a_2^2b_3b_4 + a_2^2a_3^2 - a_2^2a_4^2 + 2b_2b_3^2b_4 + 2a_4^2b_2b_3 - 2a_3^2b_2b_3)$$

$$a_{46} = a_4(2b_1^2b_4a_2^2b_2 - 4b_1a_1^2a_2^2b_4 + 2a_1^2a_2^a4^2 - 2a_1^2a_2^2a_3^2 - 2a_1^2b_4b_2a_4^2 - 2a_1^2b_3b_2b_4^2 - 2a_1^2a_4^2b_2b_3 + 2a_1^2a_3^2b_2b_3 \\ +2a_2^2b_1^2b_3b_4 + 2a_1^2b_1a_3^2b_3 + 4a_1^2a_2^2b_3b_4 - 2a_1^2b_2b_3^2b_4 - 2a_1^2a_4^2b_1b_3 - 2a_1^2b_1b_3^2b_4 + 2a_1^2b_4a_3^2b_2 \\ -2b_1a_1^2a_4^2b_4 + 2b_1a_1^2a_3^2b_4 - 2b_1b_3a_1^2b_4^2 + b_1^2a_2^2b_4^2 + 2a_1^2a_2^2b_4^2 - b_1^2a_2^2a_3^2 + b_1^2a_2^2a_4^2)$$

$$a_{47} = -a_4(2b_1^2b_4a_2^2b_2 - 2b_1a_1^2a_2^2b_4 + 2a_1^2a_2^2a_4^2 + a_1^4b_4^2 - 2a_1^2a_2^2a_3^2 + 2a_2^2b_1^2b_3b_4 + 2a_1^2a_2^2b_3b_4 + a_4^2b_1^2b_2^2 \\ +2b_1^2a_2^2b_4^2 + 2a_1^2a_2^2b_4^2 - 2b_1a_1^2b_2b_4^2 - 2b_1a_4^2a_1^2b_2 - b_1^2a_3^2b_2^2 - 2b_1^2a_2^2a_3^2 + 2b_1^2a_2^2a_4^2 + b_1^2b_2^2b_4^2 \\ -a_1^4a_3^2 + a_1^4a_4^2 + 2b_1a_1^2a_3^2b_2)$$

$$a_{48} = -a_4(2b_1a_1^2a_2^2b_3 + 2a_1^2a_2^2a_4^2 + 2a_1^2a_2^2a_3^2 + a_1^4b_3^2 + b_1^2b_2^2b_3^2 + a_4^2b_1^2b_2^2 - 2b_1a_4^2a_1^2b_2 + 3b_1^2a_3^2b_2^2 \\ +b_1^2a_2^4 + 2b_1^2a_2^2a_3^2 + 2b_1^2a_2^2a_4^2 - 2b_1^2a_2^2b_3b_2 - 2b_1a_1^2b_2b_3^2 + 3a_1^4a_3^2 + a_1^4a_4^2 - 6b_1a_1^2a_3^2b_2)$$

$$a_{56} = -2a_1^2(b_1b_3^2b_4^2 - 2b_1b_4a_3^2b_3 + 2b_1b_4a_4^2b_3 + b_1a_4^4 - 2b_1a_3^2a_4^2 + b_1a_3^4 + b_2b_3^2b_4^2 + 2b_4b_3b_2a_4^2 - 2b_4b_3b_2a_3^2 \\ +a_4^4b_2 - 2b_2a_4^2a_3^2 + b_2a_3^4)$$

$$a_{57} = -2a_1^2(4b_2a_4^2a_3^2 - 2b_1a_4^4 - 2b_1a_3^4 - 2a_4^4b_2 - a_2^2b_3b_4^2 + 2b_4b_3b_2a_3^2 - 2b_4b_3b_2a_4^2 - 2b_2a_3^4 - \\ 2b_1b_4a_4^2b_3 + 4b_1a_3^2a_4^2 + 2b_1b_4a_3^2b_3 + b_1a_2^2b_4^2)$$

$$a_{58} = -2a_1^2(2b_1a_3^4 - 2b_1a_4^4 - 2a_4^4b_2 - 2b_4b_3b_2a_3^2 - 2b_4b_3b_2a_4^2 + 2b_2a_3^4 + a_2^2a_3^2b_4 - 2b_1b_4a_4^2b_3 - \\ 2b_1b_4a_3^2b_3 - b_1a_2^2a_4^2 + a_2^2a_4^2b_4 + b_1a_2^2a_3^2)$$

$$a_{67} = -4a_1^2b_2a_4^4 - 4a_1^2a_3^4b_2 - 4b_3a_1^2a_2^2b_4^2 + 8a_1^2a_3^2b_2a_4^2 - 4a_1^2b_1a_4^4 - 4a_1^2b_1a_3^4 + 4a_1^2b_1a_2^2b_4^2 - 2a_2^2b_1^2b_3b_4^2 \\ -2a_2^2b_1^2b_4^2b_2 + 8a_1^2b_1a_3^2a_4^2 + 4a_1^2b_1b_4a_3^2b_3 - 4a_1^2b_1b_4a_4^2b_3 - 4b_3b_4a_1^2b_2a_4^2 + 4b_3b_4a_1^2a_3^2b_2$$

$$a_{68} = -4b_3b_4a_1^2a_3^2b_2 - 4b_3b_4a_1^2b_2a_4^2 - 4a_1^2b_1b_4a_3^2b_3 - 4a_1^2b_1b_4a_4^2b_3 - 4a_1^2b_2a_4^4 + 4a_1^2a_3^4b_2 - 4a_1^2b_1a_4^4 \\ +4a_1^2b_1a_3^4 + 4a_1^2a_2^2a_3^2b_4 + 4a_1^2a_2^2a_4^2b_4 + 4b_1a_1^2a_2^2a_3^2 - 4b_1a_1^2a_2^2a_4^2 + 2b_1^2a_2^2a_4^2b_4 \\ +2b_1^2a_2^2a_3^2b_4 + 2b_1^2a_2^2b_2a_4^2 - 2b_1^2a_2^2b_2a_3^2$$

$$
\begin{aligned}
a_{78} = {} & 2a_4^2 b_3 a_1^4 - 2a_1^4 a_4^2 b_4 - 4a_1^2 a_2^2 a_3^2 b_4 - 4a_1^2 a_2^2 a_4^2 b_4 + 4b_1 a_1^2 a_3^2 b_2 b_3 - 4b_1 a_1^2 a_4^2 b_2 b_3 + 4b_1 b_4 a_1^2 a_3^2 b_2 \\
& -4b_1 a_1^2 a_2^2 a_3^2 + 4b_1 a_1^2 a_2^2 a_4^2 - 2b_1^2 a_3^2 b_3 b_2^2 - 2b_1^2 b_4 b_2^2 a_3^2 - 2b_1^2 b_4 b_2^2 a_4^2 + 4b_1 b_4 a_4^2 a_1^2 b_2 - 4b_1^2 a_2^2 a_4^2 b_4 \\
& +2b_1^2 b_3 b_2^2 a_4^2 - 4b_1^2 a_2^2 a_3^2 b_4 - 4b_1^2 a_2^2 b_2 a_4^2 + 4b_1^2 a_2^2 b_2 a_3^2 - 2a_3^2 b_3 a_1^4 - 2a_1^4 a_3^2 b_4 \ .
\end{aligned}
$$

The Poisson tensor of π_{-1} is given by the formula

$$\pi_{-1} = \frac{1}{2\det L} A \ ,$$

$$\det L = P_4^2 = (a_2^2 b_1 b_4 + a_3^2 b_1 b_2 - a_4^2 b_1 b_2 - b_1 b_2 b_3 b_4 - a_1^2 a_3^2 + a_1^2 a_4^2 + a_1^2 b_3 b_4)^2 \ .$$

We note that

$$\det \pi_3 = \det \pi_1 (\det L)^2 = P_4^4 \ ,$$

and therefore

$$\det \mathcal{R} = (\det L)^2 \ .$$

This formula (as well as the formulas in the previous two cases) indicates that the eigenvalues of the recursion operator should be the squares of the non–zero eigenvalues of the Jacobi matrix. This is known to hold in the case of the classical A_n Toda lattice, see [26]. For a general recursion operator, the relation between its eigenvalues and the eigenvalues of the Lax matrix has not been fully investigated yet.
As in the case of B_2 and C_3 we have

$$\pi_1 \nabla H_2 = \pi_{-1} \nabla H_4 \ .$$

10 Conclusion

10.1 Summary of results

The classical, finite, non–periodic Toda lattice is known to be bi–Hamiltonian. Moreover, (53) is a multi–Hamiltonian formulation of the system. We have indicated how to obtain similar results for the other classical Lie algebras and we have illustrated with some small dimensional examples. These examples may be generalized:

Theorem 16 *The B_n, C_n and D_n Toda systems are bi–Hamiltonian. In fact, they are multi–Hamiltonian. In each case we define*

$$\mathcal{N} = \pi_1 \pi_3^{-1} \ ,$$

where π_1 is the Lie–Poisson bracket, π_3 is the cubic Poisson bracket and

$$\pi_{-(2i-1)} = \mathcal{N}^i \pi_1 \quad i = 1, 2, \dots \ .$$

Then all the brackets are mutually compatible, Poisson and satisfy

$$\pi_1 \nabla H_2 = \pi_{-1} \nabla H_4 = \pi_{-3} \nabla H_6 = \dots \ . \tag{150}$$

Proof: The proof of (150) is trivial. They are just the Lenard relations for the negative hierarchy. The brackets $\pi_{-1}, \pi_{-3}, \pi_{-5}, \ldots$ are all Poisson since they are generated by the negative recursion operator, $\mathcal{N}$, applied to the initial Poisson bracket π_1. To prove compatibility of all Poisson brackets appearing in (150) we take two brackets π_t and π_s where t, s are odd integers, with $t < s \leq 1$. Using condition (b) of Oevel's theorem (for the negative operator) we can express π_t as the Lie derivative of π_s in the direction of a master symmetry. It is therefore enough to prove the following simple general result: If π and σ are both Poisson tensors and $\sigma = \mathcal{L}_X \pi$ for some vector field X, then π and σ are compatible. The one line proof uses the super–Jacobi identity for the Scouten bracket:

$$[\pi, \sigma] = [\pi, [\pi, X]] = -\frac{1}{2}[X, [\pi, \pi]] = 0 \ .$$

■

We remark that Oevel's theorem applies in all three cases (and for both hierarchies) since the Euler vector field X_0 (58) is a conformal symmetry for π_1, π_3 and H_2. Furthermore, the compatibility condition holds for all brackets in both hierarchies. If π_t and π_s are both in the positive hierarchy then the argument of the theorem still works using the positive recursion operator. The remaining case, when one of the tensors has negative index and the other one positive, can also be proved in a straightforward manner using similar arguments, i.e., properties of the Schouten bracket and the fact that the formulas in Oevel's theorem hold for any integer value of the index (Theorem 8). The tensor in the positive hierarchy is the Lie derivative of the tensor in the negative hierarchy in the direction of a suitable master symmetry and the argument of the theorem shows that they are compatible. Therefore, we have a more general result: Any two brackets in either the positive or negative hierarchy are compatible.

Remark The compatibility condition follows also from a general result of bi–Hamiltonian geometry: If π and σ are two compatible Poisson tensors and π is invertible, then $\mathcal{N} = \sigma\pi^{-1}$ is a recursion operator (i.e. its torsion vanishes) and all the tensors $\mathcal{N}^i\pi$, with $i \geq 0$, are Poisson and compatible. Using this result one can prove compatibility of all brackets in both hierarchies. For example, to show that π_5 is compatible with π_{-3} we use the fact that $\mathcal{R} = \pi_3\pi_1^{-1} = \pi_{-1}(\pi_{-3})^{-1}$ to obtain $\pi_5 = \mathcal{R}^4\pi_{-3}$. Since π_{-1} and π_{-3} are compatible and Poisson, then $\mathcal{R}$ generates a chain of compatible Poisson tensors. In particular π_5 and π_{-3} are compatible.

10.2 Open problems

We conclude with some open problems and some possible directions of research for systems related to the Toda lattice.

- **Exceptional Toda lattices**

  The case of exceptional simple Lie groups is still an open problem. It is also a much more difficult problem. The only case that is reasonable to complete is the Toda system of type G_2. In that case the second Poisson bracket should be a homogeneous bracket of degree 5 (a conjecture of Flaschka states that the degrees of the independent Poisson tensors coincide with the exponents of the corresponding Lie group). The other exceptional cases are even more complicated. It is a nontrivial task even to write down an explicit Lax pair for the systems and therefore the methods of this paper will be difficult to apply.

- **Full–Kostant Toda**

  One has a tri–Hamiltonian formulation of the A_n system but no hierarchy. In the case of generalized full–Kostant Toda lattice (associated to simple Lie groups) one could seek to find similar structures as in the present paper. So far, nothing is known. The interesting feature of these systems is the presence of the rational integrals that are necessary to prove integrability. The Lie–algebraic background of the systems will certainly play a prominent role.

- **The Volterra or KM–system** The multi–Hamiltonian structure of this system was first obtained in [64]. However, there is a symplectic realization of the system, due to Volterra, and it would be interesting to find a recursion operator in that symplectic space that projects onto the known hierarchy (as in section 5.1).

- **Bogoyavlensky–Volterra lattices**

  There is also an interesting connection with the corresponding generalized Volterra systems also defined by Bogoyavlensky [65] in 1988. It seems that the multiple Hamiltonian structures of the Volterra and Toda lattices are in one–to–one correspondence through a procedure of Moser. Multiple Hamiltonian structures for the generalized Volterra lattices, were constructed recently by Kouzaris [66], at least for the classical Lie algebras. The relation between the Volterra systems of type B_n and C_n and the corresponding Toda B_n, C_n systems was demonstrated in [67]. The connection between Volterra D_n and Toda D_n is still an open problem.

- **Independence of Poisson structures** Equations (53) and (150) are remarkable; one Hamiltonian system, infinite formulations. On the other hand the systems are finite dimensional and after some point some dependencies should occur. Indeed, the integrals H_k are not all independent. It is natural to ask a similar question about the infinite sequence of Poisson structures. Do they become dependent after a certain point? Unfortunately, there exists no widely accepted definition of independence for Poisson tensors.

- **Is the Toda lattice super–integrable?** This conjecture should be true. A number of well–known systems are super–integrable, i.e. the free particle, the harmonic oscillator and the Calogero–Moser systems. In the case of the open Toda lattice, asymptotically the particles become free as time goes to infinity with asymptotic momenta being the eigenvalues of the Lax matrix. Therefore, the system behaves asympotically like a system of free particles which is super–integrable.

Acknowledgments I would like to thank H. Flaschka for introducing me to the problems of the present paper and for his considerable imput in this long project. I thank also a number of people who had discussed the subject with me and given me some useful ideas: J. M. Costa Nunes, R. Fernades, M. Gekhtman, S. Kouzaris, F. Magri, I. Marshall, W. Oevel, T. Ratiu, C. Sophocleous.

References

[1] O. I. Bogoyavlensky, Comm. Math. Phys. **51** (1976), 201.

[2] B. Kostant, Adv. Math. **34** (1979), 195.

[3] M. A. Olshanetsky and A. M. Perelomov, Invent. Math. **54** (1979), 261.

[4] M. Adler and P. van Moerbeke, Adv. in Math. **38** (1980), 267.

[5] V. V. Kozlov and D. V. Treshchev, Math. USSR-Izv. **34** (1990), 555.

[6] M. F. Ranada, J. Math. Phys. **36** (1995), 6846.

[7] A. Annamalai and K. M. Tamizhmani, J. Math. Phys. **34** (1993), 1876.

[8] A. M. Perelomov, Integrable systems of classical mechanics and Lie algebras, Vol. I, Birkhauser Verlag, Basel, 1990.

[9] P. A. Damianou, Lett. Math. Phys. **20** (1990), 101.

[10] P. A. Damianou, J. Math. Phys. **35** (1994), 5511.

[11] J. M. Nunes da Costa and P. A. Damianou, Bull. Sci. math. **125** (2001), 49.

[12] P. A. Damianou, Regul. Chaotic dyn. **5** (2000), 17.

[13] P. A. Damianou and S. P. Kouzaris, J. Phys. A. **36** (2003), 1385.

[14] P. A. Damianou, Nonlinearity **17** (2004), 397.

[15] P. A. Damianou, J. Geom. Phys. **45** (2003), 184.

[16] A. Lichnerowicz, J. Diff. Geom. **12** (1977), 253.

[17] J. E. Marsden and T. S. Ratiu, Introduction to mechanics and symmetry, A basic exposition of classical mechanical systems, Springer-Verlag, New York, 1999.

[18] I. Vaisman, Lectures on the geometry of Poisson manifolds, Progress in mathematics, **118**, Birkháuser, Basel, 1994.

[19] A. Weinstein, J. Diff. Geom. **18** (1983), 523.

[20] J. Grabowski, G. Marmo and A. M. Perelomov, Modern Phys. Lett. A **8** (1993), 1719.

[21] R. Cushman and M. Roberts, Bull. Sci. math. **126** (2002), 525.

[22] P. A. Damianou, Bull. Sci. math. **120** (1996), 195.

[23] C. Chevalley and S. Eilenberg, Trans. of Amer. Math. Soc. **63** (1948), 85.

[24] J. L. Koszul, Soc. Math. France Asterisque hors serie (1985), 257.

[25] F. Magri, J. Math. Phys. **19** (1978), 1156.

[26] G. Falqui, F. Magri and M.Pedroni, J. Nonlinear Math. Phys. **8** (2001), 118.

[27] I. M. Gelfand and I. Zakharevich, Selecta Math. **6** (2000), 131.

[28] R. G. Smirnov, C. R. Math. Rep. Acad. Sci. Canada **17** (1995), 225.

[29] Y. B. Suris, Phys. Lett. A **180** (1993), 419.

[30] B. Fuchssteiner, Progr. Theor. Phys. **70** (1983), 1508.

[31] A. S. Fokas and B. Fuchssteiner, Phys. Lett. A **86** (1981), 341.

[32] W. Oevel, *Topics in Soliton Theory and Exactly Solvable non-linear Equations*, World Scientific Publ., Singapore, 1987.

[33] H. Flaschka H, Phys. Rev. B **9** (1974), 1924.

[34] H. Flaschka, Progr. Theor. Phys. **51** (1974), 703.

[35] M. Henon, Phys. Rev. B **9** (1974), 1921.

[36] S. Manakov, Zh. Exp. Teor. Fiz. **67** (1974), 543.

[37] J. Moser, Lect. Notes Phys. **38** (1976), 97.

[38] J. Moser, Adv. Math. **16** (1975), 197.

[39] M. Toda, J. Phys. Soc. Japan **22** (1967), 431.

[40] M. Adler, Invent. Math. **50** (1979), 219.

[41] B. Kupershmidt, Asterisque **123** (1985), 1.

[42] A. Das and S. Okubo, Ann. Phys. **190** (1989), 215.

[43] R. L. Fernandes, J. Phys. A **26** (1993), 3797.

[44] C. Morosi and G. Tondo, Inv. Probl. **6** (1990), 557.

[45] P. J. Olver, J. Math. Phys. **18** (1977), 1212.

[46] L. Faybusovich and M. Gekhtman, Phys. Lett. A **272** (2000), 236.

[47] M. F. Atiyah and N. Hitchin, The geometry and dynamics of magnetic monopoles. M. B. Porter Lectures, Princeton University Press, Princeton, 1988.

[48] J. Moser, Finitely many mass points on the line under the influence of an exponential potential. Batelles Recontres, Springer Notes in Physics, 417–497 (1974).

[49] K. L. Vaninsky, J. Geom. Phys. **46** (2003), 283.

[50] F. Petalidou, Bull. Sci. math. **124** (2000), 255.

[51] Y. Kosmann-Schwarzbach and F. Magri, J. Math. Phys. **37** (1996), 6173.

[52] P. J. Olver, Applications of Lie groups to Differential Equations. GTM, **107**, Springer-Verlag, New York 1986.

[53] G. W. Bluman and S. Kumei, Symmetries and Differential Equations, Springer-Verlag, New York 1989.

[54] L. V. Ovsiannikov, Group Analysis of Differential Equations, Academic Press, New York 1982.

[55] N. H. Ibragimov, Elementary Lie group analysis and ordinary differential equations, Wiley Series in Mathematical Methods in Practice, 4, John Wiley and Sons, Ltd., 1999.

[56] J. M. Nunes da Costa and C. M. Marle, J. Phys. A **30** (1997), 7551.

[57] P. A. Damianou and C. Sophocleous, J. Math. Phys. **40** (1999), 210.

[58] P. A. Damianou, J. Phys. A **26** (1993), 3791.

[59] M. F. Ranada, J. Math. Phys. **40** (1999), 236.

[60] C. Sophocleous, S. Moyo, P.G.L. Leach, P. A. Damianou, Noether Symmetries and Integrals in One, Two and Three dimensions, TR/16/2000, Department of Mathematics and Statistics, University of Cyprus.

[61] P. A. Damianou, C. Sophocleous, Master and Noether symmetries for the Toda lattice, Proceedings of the 16th International Symposium on Nonlinear Acoustics, **1**, 2003, pp. 618–622.

[62] D. H. Collingwood and W. M. McGovern, Nilpotent orbits in semisimple Lie algebras, Van Nostrand Reinhold Co., New York, 1993.

[63] J. E. Humphreys, Introduction to Lie algebras and representation theory GTM **9**, Springer–Verlag 1972.

[64] P. A. Damianou, Phys. Lett. A **155** (1991), 126.

[65] O. I. Bogoyavlensky, Phys. Lett. A. **134** (1988), 34.

[66] S. P. Kouzaris, J. Nonlinear Math. Phys. **10** (2003), 431.

[67] P. A. Damianou and R. Fernandes, Rep. Math. Phys. **50** (2002), 361.